Continuous multiplicative isometries of a matrix subalgebra

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Notation

- $M_n(\mathbb{F})$ is the space of $n \times n$ matrices ($n \geq 2$) over a field $\mathbb{F}$.
- A map $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is *multiplicative* if

  $$\psi(AB) = \psi(A)\psi(B)$$

  for all $A, B \in M_n(\mathbb{F})$.
- Given an endomorphism f of the field $\mathbb{F}$, A_f is the matrix whose (i, j)-entry is $f(A_{ij})$.

Examples of multiplicative maps on M_n

- $f : \mathbb{F} \to \mathbb{F}$ is an endomorphism; $X \mapsto X_f$
- $S \in M_n(\mathbb{F})$ is invertible; $X \mapsto SXS^{-1}$
- every singular matrix $\mapsto 0$; group homomorphism on $GL_n(\mathbb{F})$

A map $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is said to be *degenerate* if ψ maps every singular matrix to 0.

Multiplicative maps on M_n

Theorem (Jodeit & Lam 1969)

Suppose that $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is multiplicative. Then either

1. *ψ is degenerate, or*
2. *$\psi(A) = \phi(A) + P$ for some nonzero idempotent P and degenerate multiplicative map ϕ, or*
3. *$\psi(A) = SA_f S^{-1}$ for some invertible $S \in M_n(\mathbb{F})$ and endomorphism $f : \mathbb{F} \to \mathbb{F}$, or*
4. *$\psi(A) = S(\operatorname{adj} A_f)^t S^{-1}$ for some invertible $S \in M_n(\mathbb{F})$ and endomorphism $f : \mathbb{F} \to \mathbb{F}$.*

Result holds when $\mathbb{F}$ is just a principal ideal domain.

Multiplicative isometries on M_n

Corollary

Let $n \geq 2$. Suppose that $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a multiplicative isometry. Then

$$\psi(A) = SA_f S^{-1}$$

for some invertible $S \in M_n$ and endomorphism $f : \mathbb{C} \to \mathbb{C}$.

Multiplicative isometries on M_1

Let $k \in \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be an additive map. Then using the polar decomposition $re^{i\theta}$ for complex numbers, the map

$$\psi(re^{i\theta}) = \begin{cases} re^{ik \ln r} e^{ig(\theta)} & \text{if } r \neq 0 \\ 0 & \text{if } r = 0 \end{cases}$$

is a multiplicative isometry on $M_1(\mathbb{C})$.

The spectral norm of $A \in M_n(\mathbb{C})$ is

$$\|A\| = \max\{(x^*A^*Ax)^{1/2} : x \in \mathbb{C}^n, x^*x = 1\}$$

Theorem (Cheung, Fallat, Li 2002)

Suppose $\mathcal{S}$ is a semigroup of $M_n(\mathbb{C})$ containing all rank-1 matrices. Then a multiplicative map $\psi : \mathcal{S} \to M_n(\mathbb{C})$ is a spectral norm isometry if and only if there is a unitary U such that

- $\psi(A) = UAU^*$ *for all* $A \in \mathcal{S}$, *or*
- $\psi(A) = U\overline{A}U^*$ *for all* $A \in \mathcal{S}$.

The numerical radius of $A \in M_n(\mathbb{C})$ is

$$w(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\}$$

Theorem (Cheung, Fallat, Li 2002)

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Other semigroups?

What if the semigroup S has very few rank-1 matrices?

Let $\mathcal{A}$ be the unital abelian algebra of $n \times n$ upper-triangular Toeplitz matrices.

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ 0 & \dots & 0 & a_1 \end{bmatrix}$$

$\mathcal{A}$ contains effectively one rank-1 matrix; its only idempotents are trivial.

Spectral norm isometries on $\mathcal{A}$

Theorem (Farenick, Mastnak, Popov 2016)

Let $\psi : \mathcal{A} \to M_n(\mathbb{C})$ be a continuous multiplicative map that preserves the spectral norm. Then there exists a unitary U such that either

- $\phi(A) = UAU^*$ *for all* $A \in \mathcal{A}$*, or*
- $\phi(A) = U\bar{A}U^*$ *for all* $A \in \mathcal{A}$*.*

Proof uses the isometry condition at the outset!

Continuous multiplicative maps on $\mathcal{A}$

Let $\mathcal{A}$ be the unital abelian algebra of $n \times n$ upper-triangular Toeplitz matrices.

Theorem

Suppose $\psi : \mathcal{A} \to M_n(\mathbb{C})$ is a continuous multiplicative map for which $\psi(X) = \psi(0)$ implies $X = 0$. Then there exist an invertible $S \in M_n(\mathbb{C})$ and $\gamma \in \mathbb{C}$ such that

$$\psi(A) = S^{-1}\phi(A)^{+}S$$

for all $A \in \mathcal{A}$. Here X^{+} is X or $\overline{X}$, and

$$\phi(A) = A + \gamma \frac{Tr\,A}{n} \ln\left|\frac{Tr\,A}{n}\right| E_{1n},$$

where $0 \ln 0 = 0$.

Sketch of the proof

- $\psi(I)$ is idempotent; we may write

$$\psi(I) = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

- $\psi(0)$ is idempotent; we may write

$$\psi(0) = \begin{bmatrix} I_q & 0 & 0 \\ 0 & 0_{r-q} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Every $\psi(A)$ has the form

$$\psi(A) = \begin{bmatrix} I_q & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Let J be the matrix with 1s on the superdiagonal and 0s elsewhere. Write

$$\psi(J) = \begin{bmatrix} I_q & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $\psi(0) = \psi(J^n) = \psi(J)^n$, so B is nilpotent.

- $B^{r-q} = 0 \implies \psi(J^{r-q}) = \psi(0) \implies J^{r-q} = 0$

- Thus $r = n$ and $q = 0$, that is, $\psi(I) = I$ and $\psi(0) = 0$.

- $\psi(J)$ is nilpotent of index n; after applying a similarity we may conclude $\psi(J^k) = J^k$ for all k.

Hard part

- Since $\psi(\mathcal{A}) \subseteq \mathcal{A}$, we may write

  $$\psi(z_0 I + z_1 J + \cdots + z_{n-1} J^{n-1}) = f_0(\vec{z}) I + f_1(\vec{z}) J + \cdots + f_{n-1}(\vec{z}) J^{n-1},$$

  where $\vec{z} = (z_0, \ldots, z_{n-1}) \in \mathbb{C}^n$.

- Solve functional equations in $f_0, \ldots, f_{n-1}$.
  - Let S be the forward shift, that is

    $$S\vec{z} = (0, z_0, \ldots, z_{n-2})$$

    Then

    $$f_i(\vec{z}) = f_{i+k}(S^k \vec{z})$$

    for $k = 0, \ldots, n-1-i$.
  - Thus f_i depends on only the first $i+1$ variables $z_0, \ldots, z_i$.

- Continuity turns additive functions into $\mathbb{R}$-linear functions.
- Solve for f_0, f_1.
- Induction hypothesis:

  $$f_i(\vec{z}) = z_i \quad \forall i = 0, \ldots, r-2 \quad \text{and} \quad f_{r-1}(\vec{z}) = z_{r-1} + cz_0 \ln |z_0|$$

  Induct on r.
- Compute and perform another induction to show that f_r does not depend on the variables $z_{r-1}, \ldots, z_1$. $\square$

Without continuity ...

- Let $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Let $\mathcal{A}_2$ be the algebra generated by N.
- Let $h : \mathbb{C} \to \mathbb{C}$ be additive.
- Let $f : \mathbb{C} \to \mathbb{C}$ be multiplicative, with $f(0) = 0$ and $f(1) = 1$.

Example

Define a map $T : \mathcal{A}_2 \to M_2(\mathbb{C})$ by

$$T(wI + zN) = \begin{cases} f(z)N & \text{if } w = 0 \\ f(w)(I + h(z/w)N) & \text{if } w \neq 0 \end{cases}$$

Then T is a multiplicative map fixing I and N. If f and h are one-to-one then so is T.

Continuous multiplicative isometries

Let $\mathcal{A}$ be the algebra of $n \times n$ upper-triangular Toeplitz matrices.

Theorem

Suppose $T : \mathcal{A} \to M_n$ is a continuous multiplicative isometry. Then there exists an invertible matrix S such that either

- $T(A) = S^{-1}AS$ *for all* $A \in \mathcal{A}$, *or*
- $T(A) = S^{-1}\overline{A}S$ *for all* $A \in \mathcal{A}$.

This holds for any norm, even norms that are not submultiplicative.

Proof

For any $A \in \mathcal{A}$ we have

$$T(A) = S^{-1}\left(A + \gamma \frac{\operatorname{Tr} A}{n} \ln \left|\frac{\operatorname{Tr} A}{n}\right| E_{1n}\right)^{+} S$$

for some invertible S and scalar γ. Thus for all $r > 0$,

$$r\|I\| = \|rI\| = \|T(rI)\| = \|S^{-1}(rI + (\gamma r \ln r)E_{1n})^{+} S\|$$

$$= r\|S^{-1}(I + (\gamma^{+} \ln r)E_{1n})S\| \geq r\left|\|I\| - \|(\gamma^{+} \ln r)S^{-1}E_{1n}S\|\right|$$

Divide by r and let $r \to \infty$ to conclude $\gamma = 0$. □

Let $\mathcal{A}$ be the unital algebra generated by a non-derogatory matrix $A \in M_n$. We may write

$$A = P^{-1}(J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k))P$$

for some invertible $P \in M_n$ and distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.

Theorem

Suppose $T : \mathcal{A} \to M_n$ is a continuous multiplicative isometry. Then there exist an invertible $S \in M_n$, $c_j \in \mathbb{R}$, and $m_j \in \mathbb{Z}$ for $1 \leq j \leq k$ so that, for all $X = P^{-1}(X_1 \oplus \cdots \oplus X_k)P$, we have

$$T(X) = S^{-1}(\psi_1(X_1) \oplus \cdots \oplus \psi_k(X_k))S,$$

where

$$\psi_j(A) = A \quad \text{or} \quad \psi_j(A) = \overline{A} \qquad \text{if } n_j > 1$$

(chosen independently for each j); if $n_j = 1$ then

$$\psi_j(re^{i\theta}) = re^{ic_j \ln r + im_j\theta}.$$

Sufficiency

Sufficiency for such maps to be isometries depends on both the norm and the algebra.

For instance, if $\mathcal{A}$ is the algebra of diagonal matrices and the norm is unitarily invariant, all of the listed maps are isometries if S is unitary.

On the other hand, let $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and consider

$$\mathcal{A} = \left\{ S \begin{bmatrix} w & 0 \\ 0 & z \end{bmatrix} S^{-1} : w, z \in \mathbb{C} \right\} = \left\{ \begin{bmatrix} w & z - w \\ 0 & z \end{bmatrix} : w, z \in \mathbb{C} \right\}$$

In this case $w \mapsto w$, $z \mapsto \overline{z}$ is not an isometry.

Restrict to similarity transformations

Completely characterizing isometries on a subalgebra is difficult, even if we restrict to the spectral norm and similarity transformations.

Let $\mathcal{A} \subset M_n$ be the (unital) algebra generated by a nilpotent A of index $n-1$.

When is $X \mapsto S^{-1}XS$ an isometry on $\mathcal{A}$?

- If $\mathcal{A}$ is the upper-triangular Toeplitz algebra, then S must be unitary.
- In general, if $\mathcal{A}$ is just similar to the upper-triangular Toeplitz algebra, then ?????

Reformulate

Let $\mathcal{A} \subset M_n$ be the (unital) algebra generated by a nilpotent A of index $n-1$.

When is $X \mapsto S^{-1}XS$ an isometry on $\mathcal{A}$?

Let $B = S^{-1}AS$. Equivalently, when is $\|f(B)\| = \|f(A)\|$ for all polynomials f?

Theorem (Farenick, Gerasimova, Shvai 2011)

Fix an upper triangular Toeplitz matrix A with nonzero entries on the superdiagonal.

If $\|f(A)\| = \|f(B)\|$ for all polynomials f, then B is unitarily similar to A.

This is not true for general upper triangular matrices ...

Example (Farenick, Gerasimova, Shvai 2011)

If $0 < x < y$ then

$$A = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 0 & y & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}$$

satisfy $\|f(A)\| = \|f(B)\|$ for all polynomials f, but A and B are not unitarily similar.

Note that $B = W^* A^T W$, where

$$W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Question

Suppose A and B are similar, and

$$\|f(A)\| = \|f(B)\|$$

for all polynomials f.

Must B be unitarily similar to either A or A^T?

The End

Thank you for your attention!