On Solving DC Optimization Problem over SOS-concave Matrix Polynomial Constraint

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Abstract

- In this talk, we consider a DC optimization problem (P) with the difference of a SOS-convex polynomial and a support function as its objective function and with a SOS-concave matrix polynomial constraint.
- Using a set containment characterization, we derive a zero duality gap result under the Slater's condition for the DC optimization problem (P) and its dual problem $(\mathrm{D})^{\mathrm{sos}}$. Here the dual problem $(\mathrm{D})^{\mathrm{sos}}$ of (P) can be represented by a sum of squares relaxation problem, a semidefinite optimization problem (SDD), which is equivalent to $(\mathrm{D})^{\mathrm{sos}}$, is considered and its dual problem (SDP) of (SDD) is given.

Abstract

- Also, we present the relations of the optimal solution of (P) and the optimal solution of (SDP), and the optimal solution of $(\mathrm{D})^{\mathrm{sos}}$ and (SDD). The relations show us that (SDP) is an exact SDP relaxation for (P), and (SDD) is an exact SDP relaxation for $(\mathrm{D})^{\mathrm{sos}}$.

Contents

Introduction

- Recently, many authors [1, 2, 20, 29] have investigated SOS-convex polynomials and their applications. The class of SOS-convex polynomials include separable convex polynomials and convex quadratic functions as their special cases.

[1] A.A. Ahmadi, P.A. Parrilo. A convex polynomial that is not SOS-convex, *Math. Program.*, 135 (2012).

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[29] V. Jeyakumar and J. Vicente-Pérez, V. Jeyakumar and J. Vicente-Pérez, Dual semidefinite programs without duality gaps for a class of convex minimax programs, *J. Optim. Theory Appl.*, 162 (2014).

Introduction

- The important feature of SOS-convexity, which distinguishes from convexity for polynomials is that one can numerically check whether a polynomial is SOS-convex or not by solving a related semi-definite optimization (feasibility) problem which can be solved efficiently via interior point methods [31].
- In particular, the gap between SOS-convex polynomial and convex polynomial is completely characterized in [2].

[2] A. A. Ahmadi and P. A. Parrilo, A complete characterization of the gap between convexity and SOS-convexity, *SIAM J. Optim.*, 23 (2013).

Introduction

- The exact semidefinite programming relaxation or strong duality involving dual semidefinite programs is a highly desirable property particularly because semidefinite programming can be efficiently solved (e.g. using interior point methods)

Preliminaries

Throughout this talk,

- $\mathbb{R}^n$ denotes the Euclidean space with dimension n
- For a set A in $\mathbb{R}^n$, the convex hull of A is denoted by co$A$.
- A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if for all $\mu \in [0, 1]$,

$$f((1-\mu)x + \mu y) \leq (1-\mu)f(x) + \mu f(y)$$

  for all $x, y \in \mathbb{R}^n$.
- The function f is said to be concave whenever $-f$ is convex.

Preliminaries

- We say that a real polynomial f is sum of squares if there exist real polynomials f_j, $j = 1, \ldots, r$, such that $f = \sum_{j=1}^{r} f_j^2$.
- The set consisting of all sum of squares real polynomial is denoted by Σ^2.
- Moreover, the set consisting of all sum of squares real polynomial with degree at most d is denoted by Σ_d^2.

Preliminaries

- For a multi-index $\alpha := (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, let $|\alpha| := \sum_{i=1}^n \alpha_i$. x^α denotes the monomial $x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n}$.
- Consider the vector $v_d(x) = (x^\alpha)_{|\alpha| \le d} = (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_{n-1}x_n, x_n^2, \ldots, x_1^d, \ldots, x_n^d)^T$, of all the monomials x^α of degree less than or equal to d, which has dimension $s(d) := \begin{pmatrix} n+d \\ n \end{pmatrix}$. For details see [31].
- Let S^n be the set of $n \times n$ symmetric matrices and let $S^n_+ = \{\Lambda \in S^n \mid \mathrm{tr}(\Lambda X) \ge 0 \ \forall X \in S^n\}$. For $M, N \in S^n$, $M \succeq N$(resp. $M \succ N$) if and only if $M - N$ is positive semidefinite(resp. positive definite).

[31] J. B. Lasserre, Moments, Positive Polynomials and Their Applications, Imperial College Press, 2009.

Preliminaries

We now introduce the definition of SOS-convex polynomials.

Definition 2.1. [1, 2, 20]

A real polynomial f on $\mathbb{R}^n$ is called SOS-convex if the Hessian matrix function $H : x \mapsto \nabla^2_{xx} f(x)$ is a SOS matrix polynomial, that is, there exist a matrix polynomial $F(x)$ such that
$\nabla^2_{xx} f(x) = F(x)F(x)^T$,
equivalently, for all x, $y \in \mathbb{R}^n$ and for all $\lambda \in [0, 1]$,

$$\lambda f(x) + (1 - \lambda)f(y) - f(\lambda x + (1 - \lambda)y)$$

is a sum of squares polynomial in $\mathbb{R}[x; y]$(with respect to variables x and y). Moreover, we say f is SOS-concave if $-f$ is SOS-convex.

Preliminaries

- Clearly, a SOS-convex polynomial is convex. However, the converse is not true.
- Thus, there exists a convex polynomial which is not SOS-convex [1, 2].
- The degree of a polynomial g is denoted by $\deg g$.
- Let us denote the set of convex and SOS-convex polynomials in n variables of degree d by $\tilde{C}_{n,d}$ and $\tilde{\Sigma} C_{n,d}$, respectively. Then $\tilde{C}_{n,d} = \tilde{\Sigma} C_{n,d}$ if and only if $n = 1$ or $d = 2$ or $(n, d) = (2, 4)$ [2].

Preliminaries

Now we introduce the definition of concave matrix.

Definition 2.2.

$(m \times m)$ polynomial symmetric matrix $G(x)$ is called concave if for any $x,\ y \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$,

$$G((1-\lambda)x + \lambda y) \succeq (1-\lambda)G(x) + \lambda G(y).$$

Preliminaries

Now we give the properties of concave matrix.

Remark 2.1.

Let $G(x)$ be a symmetric $(m \times m)$ polynomial matrix. Then the following statements are equivalent:

(i) $G(x)$ is concave;

(ii) For all $\Lambda \in S^m_+$, $-\langle \Lambda, G(x) \rangle$ is convex, where $\langle \Lambda, G(x) \rangle = \mathrm{tr}(\Lambda G(x))$;

(iii) For all $\xi \in \mathbb{R}^m$, $-\langle \xi\xi^T, G(x) \rangle$ is convex;

(iv) For all $\xi \in \mathbb{R}^m$, $-\xi^T G(x)\xi$ is convex;

(v) For all $\xi \in \mathbb{R}^m$, $-\nabla^2_{xx}(\xi^T G(x)\xi) \succeq 0$.

Preliminaries

The definition of SOS-concave matrix is as follows:

Definition 2.3.

A symmetric $(m \times m)$ polynomial matrix $G(x)$ is called SOS-concave matrix if for every $\xi \in \mathbb{R}^m$, there exists a polynomial matrix $F_\xi(x)$ in x such that

$$-\nabla^2_{xx}(\xi^T G(x)\xi) = F_\xi(x)^T F_\xi(x).$$

Preliminaries

Now we give the properties of SOS-concave matrix.

Remark 2.2.

Let $G(x)$ be a symmetric $(m \times m)$ polynomial matrix. Then the following statements are equivalent:

(i) $G(x)$ is SOS-concave;

(ii) For all $\Lambda \in S^m_+$, $-\langle \Lambda, G(x) \rangle$ is SOS-convex;

(iii) For all $\xi \in \mathbb{R}^m$, $-\langle \xi\xi^T, G(x) \rangle$ is SOS-convex;

(iv) For all $\xi \in \mathbb{R}^m$, $-\xi^T G(x) \xi$ is SOS-convex.

Preliminaries

The following simple example illustrates a SOS-concave matrix polynomial.

Example 2.1.

Consider a matrix polynomial

$$G(x_1, x_2) = \begin{pmatrix} -x_1^2 - 4x_1 - 3 - x_2^2 & x_2 \\ x_2 & -x_2 \end{pmatrix}.$$

Then, for all $\xi = (\xi_1, \xi_2)^T \in \mathbb{R}^2$,

$$-\xi^T G(x_1, x_2)\xi = \xi_1^2 x_1^2 + \xi_1^2 x_2^2 + 4\xi_1^2 x_1 + (\xi_2^2 - 2\xi_1\xi_2)x_2 + 3\xi_1^2.$$

and

Preliminaries

Example 2.1. cont.

$$-\nabla^2_{xx}(\xi^T G(x)\xi) = \begin{pmatrix} 2\xi_1^2 & 0 \\ 0 & 2\xi_1^2 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\xi_1 & 0 \\ 0 & \sqrt{2}\xi_1 \end{pmatrix}^T \begin{pmatrix} \sqrt{2}\xi_1 & 0 \\ 0 & \sqrt{2}\xi_1 \end{pmatrix}.$$

So, $-\xi^T G(x)\xi$ is a SOS-convex polynomial. It follows from Remark 2.2 (iv) that $G(x)$ is a SOS-concave matrix polynomial.

Preliminaries

We now introduce how to recognize whether a polynomial can be written as a sum of squares via positive semidefinite programming.

Proposition 2.1. [31]

A polynomial $g \in \mathbb{R}[x]_{2d}$ has a sum of squares decomposition if and only if there exists a real symmetric and positive semidefinite matrix $Q \in \mathbb{R}^{s(d)\times s(d)}$ such that $g(x) = v_d(x)^T Q v_d(x)$, for all $x \in \mathbb{R}^n$.

Now we let $v_d(x)v_d(x)^T = \sum_{\alpha\in\mathbb{N}^n} x^\alpha B_\alpha$, where B_α are $s(d) \times s(d)$ real symmetric matrices, Then $g(x) = \sum_{\alpha\in\mathbb{N}^n} g_\alpha x^\alpha$ is a sum of squares if and only if solving the following semidefinite feasibility problem [31]:

$$\text{Find } Q \in \mathbb{R}^{s(d)\times s(d)} \text{ such that}$$
$$Q = Q^T,\ Q \succeq 0,\ \langle Q, B_\alpha\rangle = g_\alpha,\ \forall\alpha \in \mathbb{N}^n.$$

Set Containment Characterization

Under Slater's condition, we can obtain the following set containment result:

Theorem 3.1.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a SOS-convex polynomial and let G(x) be a $(m \times m)$ symmetric SOS-concave matrix polynomial. Let $h(x) = \max_{u \in \text{co}\{v_1,\ldots,v_l\}} u^T x$, where $v_1, \ldots, v_l \in \mathbb{R}^n$. Assume that $K := \{x \in \mathbb{R}^n : G(x) \succeq 0\} \neq \emptyset$. Assume that there exists $\hat{x} \in \mathbb{R}^n$ such that $G(\hat{x}) \succ 0$. Then the following statements are equivalent:

(i) $\{x \in \mathbb{R}^n : G(x) \succeq 0\} \subset \{x \in \mathbb{R}^n : f(x) - h(x) \geq 0\}$;

(ii) For each $i = 1, \ldots, l$, there exist $\Lambda_i \in S^m_+$ such that

$$f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot) \rangle \in \Sigma^2.$$

Exact SDP Relaxations

The following DC optimization problem is our problem which is treated in this talk:

$$
\begin{array}{lll}
(\mathrm{P}) & \inf & \{f(x) - h(x)\} \\
 & \mathrm{s.t.} & G(x) \succeq 0,
\end{array}
$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a SOS-convex polynomial, $G(x)$ is a symmetric $(m \times m)$ SOS-concave matrix polynomial and $h(x) = \max_{u \in \mathrm{co}\{v_1, \ldots, v_l\}} u^T x$, for $v_1, \ldots, v_l \in \mathbb{R}^n$. Let $K := \{x \in \mathbb{R}^n : G(x) \succeq 0\}$.

In sequel, we assume that the optimal value of (P) is finite. Moreover, the problem (P) can be rewritten as follows:

$$\min_{i=1,\ldots,l} \inf_{x \in K} \{f(x) - v_i^T x\}.$$

Exact SDP Relaxations

The Lagrangian dual problem for (P) is given by

$$(\mathrm{LD}) \quad \min_{i=1,\ldots,l} \sup_{\Lambda \in S_+^m} \inf_{x \in \mathbb{R}^n} \{ f(x) - v_i^T x - \langle \Lambda_i, G(x) \rangle \}.$$

which can equivalently be written as

$$(\mathrm{D}) \quad \min_{i=1,\ldots,l} \sup_{\mu_i \in \mathbb{R}, \Lambda \in S_+^m} \{ \mu_i \in \mathbb{R} \mid f(x) - v_i^T x - \langle \Lambda_i, G(x) \rangle - \mu_i \geq 0, \ \forall x \in \mathbb{R}^n \}$$

A sum of squares relaxation problem of (D) is as follows :

$$(\mathrm{D})^{\mathrm{sos}} \quad \min_{i=1,\ldots,l} \sup_{\mu_i \in \mathbb{R}, \Lambda_i \in S_+^m} \{ \mu_i \in \mathbb{R} \mid f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot) \rangle - \mu_i \in \Sigma^2 \}.$$

Exact SDP Relaxations

From Proposition 2.1, $f - v_i^T(\cdot) - \langle \Lambda, G(\cdot)\rangle - \mu \in \Sigma_d^2$ if and only if there exists a symmetric matrix $X \in \mathbb{R}^{s(d)\times s(d)}$ such that

$$f - v_i^T(\cdot) - \langle \Lambda, G(\cdot)\rangle - \mu = v_d(x)^T X v_d(x),$$

that is, $\sum_\alpha f_\alpha x_\alpha - \langle \Lambda, \sum_\alpha G_\alpha x_\alpha\rangle - \mu = \sum_\alpha \langle X, B_\alpha\rangle x_\alpha$. i.e., $f_0 - \langle \Lambda, G_0\rangle - \mu = 0$ and $\langle \Lambda, G_\alpha\rangle + \langle X, B_\alpha\rangle = f_\alpha - (v_i^T(\cdot))_\alpha$, $\alpha \neq 0$, where $f(x) = \sum_\alpha f_\alpha x^\alpha$, $v_i^T(\cdot) = \sum_\alpha (v_i^T(\cdot))_\alpha$, $G(x) = \sum_\alpha G_\alpha x^\alpha$ and $v_d(x)^T v_d(x) = \sum_\alpha x^\alpha B_\alpha$. Thus $(\mathrm{D})^{\mathrm{sos}}$ can be rewritten as the following semidefinite problem:

$$\begin{aligned}(\mathrm{SDD}) \quad \min_{i=1,\dots,l} \ \sup_{X,\Lambda_i} \quad & f_0 - \langle \Lambda_i, G_0\rangle - \langle X, B_0\rangle \\ \text{s.t.} \quad & \langle \Lambda_i, G_\alpha\rangle + \langle X, B_\alpha\rangle = f_\alpha - (v_i^T(\cdot))_\alpha, \\ & \alpha \neq 0,\ X \succeq 0,\ X = X^T,\ \Lambda_i \in S_+^m.\end{aligned}$$

Exact SDP Relaxations

The dual problem of (SDD) is the following semidefinite problem (SDP):

$$
\begin{aligned}
(\mathrm{SDP}) \min_{i=1,\ldots,l} \quad & \inf_{y} \quad f_0 + \sum_{\alpha \neq 0} (f - v_i^T(\cdot))_\alpha y_\alpha \\
& \text{s.t.} \quad G_0 + \sum_{\alpha \neq 0} y_\alpha G_\alpha \succeq 0, \;\; B_0 + \sum_{\alpha \neq 0} y_\alpha B_\alpha \succeq 0.
\end{aligned}
$$

(SDP) is our exact SDP relaxation for (P). (SDD) is our exact SDP relaxation for $(\mathrm{D})^{\mathrm{sos}}$.

Exact SDP Relaxations

Now, using the result of Theorem 3.1 (Set Containment Characterization), we give a zero duality gap result for (P), $(\mathrm{D})^{\mathrm{sos}}$, (SDD) and (SDP) under the Slater's condition.

Theorem 4.1. (Zero duality gap)

Assume that $\inf(\mathrm{P}) := f^*$ is finite and the Slater condition holds, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $G(\hat{x}) \succ 0$. Let $K := \{x \in \mathbb{R}^n \mid G(x) \succeq 0\} \neq \emptyset$. Then

$$\mathrm{Val}(\mathrm{P}) = \mathrm{Val}(\mathrm{D})^{\mathrm{sos}} = \mathrm{Val}(\mathrm{SDD}) = \mathrm{Val}(\mathrm{SDP}),$$

where $\mathrm{Val}(\mathrm{P})$, $\mathrm{Val}(\mathrm{D})^{\mathrm{sos}}$, $\mathrm{Val}(\mathrm{SDD})$ and $\mathrm{Val}(\mathrm{SDP})$ represent the optimal values of (P), $(\mathrm{D})^{\mathrm{sos}}$, (SDD) and (SDP), respectively.

Exact SDP Relaxations

Now, we give the relations of the optimal solution of (P) and the optimal solution of (SDP), and the optimal solution of $(\mathrm{D})^{\mathrm{sos}}$ and (SDD).

Theorem 4.2.

Assume that $\inf(\mathrm{P}) := f^*$ is finite and the Slater condition holds, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $G(\hat{x}) \succ 0$. Let $K := \{x \in \mathbb{R}^n \mid G(x) \succeq 0\} \neq \emptyset$. Then the following statements hold:

(i) $\bar{x}$ is a minimizer of (P) iff the vector $\bar{y} := (\bar{x}_1, \ldots, \bar{x}_n, \bar{x}_1^2, \bar{x}_1\bar{x}_2, \ldots, \bar{x}_1^{2d}, \ldots, \bar{x}_n^{2d})$ is a minimizer of (SDP).

(ii) $(\bar{\Lambda}_{i_0}, \bar{\mu}_{i_0}) \in S^m_+ \times \mathbb{R}$ is a maximizer of $(\mathrm{D})^{\mathrm{sos}}$ iff $(\bar{\Lambda}_{i_0}, \bar{X}) \in S^m_+ \times S^{s(d)}_+$ is a maximizer of (SDD) for some $\bar{X} = \sum_{k=1}^r \bar{q}_k^{i_0} {\bar{q}_k^{i_0}}^T$ and $\bar{q}_k^{i_0} \in \mathbb{R}^{s(d)}$.

Exact SDP Relaxations

From Theorem 4.2, we can solve (P) by solving the semidefinite program (SDP), and we can solve $(\mathrm{D})^{\mathrm{sos}}$ by solving the semidefinite program (SDD). The semidefinite programs can be solved efficiently by interior point methods. So, (SDP) is the exact SDP relaxation for (P) and (SDD) is the exact relaxation of $(\mathrm{D})^{\mathrm{sos}}$.

Exact SDP Relaxations

We can illustrate our main results (Theorem 4.1 and Theorem 4.2) by the following example:

Example 4.1.

Consider the following problem:

$$\begin{array}{lll} (\mathrm{P}_0) & \min & x_1^8 + x_1x_2 + x_1^2 + x_2^2 - |x_1| - |x_2| \\ & \text{subsect to} & \begin{pmatrix} -x_1^2 - 4x_1 - 3 - x_2^2 & x_2 \\ x_2 & -x_2 \end{pmatrix} \succeq 0. \end{array}$$

Let $f(x_1, x_2) = x_1^8 + x_1x_2 + x_1^2 + x_2^2$,
$h(x) = \max_{\substack{u_1 \in [-1,1] \\ u_2 \in [-1,1]}} (u_1, u_2)^T (x_1, x_2) = \max_{(u_1,u_2) \in \mathrm{co} M} (u_1, u_2)^T (x_1, x_2) =$
$|x_1| + |x_2|$, where $M = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$

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Thank you for your attention!