The inequality of Hadamard product for the Karcher means

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Geometric means

In 1975 W. Pusz and S. L. Woronowicz have introduced the definition of the geometric mean

$$A\#B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2} \tag{1}$$

for positive definite matrices A and B, and then T. Ando in 1979 has developed its theory.

(1) (Riccati Lemma) $A\#B$ is the unique positive definite solution of the equation $XA^{-1}X = B$.

(2) (Ando) $(A\#B) \circ (A\#B) \leqslant A \circ B$, where $A \circ B = [a_{ij}b_{ij}]$ is the Hadamard product for $A = [a_{ij}]$ and $B = [b_{ij}]$. Here, the relation $\leqslant$ is the Löewner order defined as

$$A \leqslant B \text{ if and only if } B - A \text{ is positive semidefinite.}$$

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Geometric means

Let $\mathbb{P}$ be the open convex cone of positive definite matrices equipped with the Riemannian trace metric δ:

$$\delta(A, B) = \| \log(A^{-1/2}BA^{-1/2})\|_2.$$

The unique Riemannian geodesic curve in $\mathbb{P}$ connecting from A to B is given by

$$t \in [0, 1] \mapsto A\#_t B := A^{1/2} \left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}, \tag{2}$$

called the weighted geometric mean of A and B. It satisfies

$$(A\#_s B)\#_u(A\#_t B) = A\#_{(1-u)s+ut}B \tag{3}$$

for any $s, t, u \in [0, 1]$.

Sagae-Tanabe inductive means

There have been introduced many different kinds of generalizing two-variable geometric mean to multi-variable geometric means, such as the Sagae-Tanabe inductive mean, the Ando-Li-Mathias symmetrization procedure, and the Bini-Meini-Poloni method, etc.

For given n-tuple of positive definite matrices $\mathbb{A} = (A_1, A_2, \ldots, A_n)$ and a positive probability vector $\omega = (w_1, \ldots, w_n)$, the Sagae-Tanabe inductive mean $G_\omega(\mathbb{A})$ is defined as :

$$G_\omega(\mathbb{A}) := A_n \#_{\alpha_{n-1}} (A_{n-1} \#_{\alpha_{n-2}} \cdots \#_{\alpha_2} (A_2 \#_{\alpha_1} A_1)),$$

where for $1 \leqslant k \leqslant n-1$

$$\alpha_k = 1 - w_{k+1} \left(\sum_{j=1}^{k+1} w_j \right)^{-1}.$$

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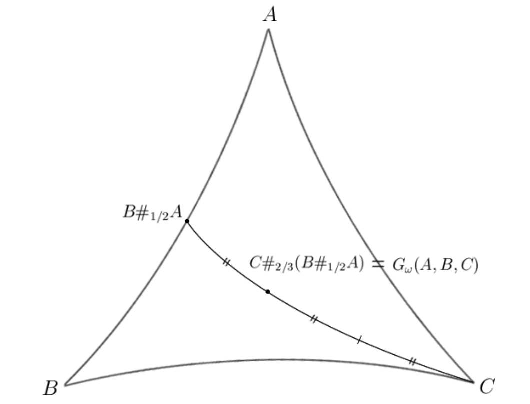
Let $\mathbb{A} = (A, B, C) \in \mathbb{P}^3$ and $\omega = (1/3, 1/3, 1/3)$.



Sagae-Tanabe inductive means

B. Feng and A. Tonge in 2005 have extended Ando's inequality for Hadamard product to the Sagae-Tanabe mean $G_\omega(\mathbb{A})$.

Theorem

Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$ *and let* ω *be a positive probability vector in* $\mathbb{R}^n$*. If* $\sigma_1, \sigma_2, \ldots, \sigma_n$ *are permutations on* $\{1, 2, \ldots, n\}$ *satisfying*

$$\{\sigma_1(j), \sigma_2(j), \ldots, \sigma_n(j)\} = \{1, 2, \ldots, n\}$$

for each $j = 1, 2, \ldots, n$*, then*

$$G_\omega(\mathbb{A}_{\sigma_1}) \circ \cdots \circ G_\omega(\mathbb{A}_{\sigma_n}) \leqslant A_1 \circ \cdots \circ A_n, \tag{4}$$

where $\mathbb{A}_{\sigma_k} = (A_{\sigma_k(1)}, \ldots, A_{\sigma_k(n)}) \in \mathbb{P}^n$*.*

The Karcher means

Note that $(\mathbb{P}, \delta)$ is a Hadamard space (or a non-positive curvature space), which is a complete metric space satisfying the semi-parallelogram law.

For an n-tuple $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$ and a positive probability vector $\omega = (w_1, \ldots, w_n)$, the minimizer of the weighted sum of squares of the Riemannian distances to each of $A_1, \ldots, A_n$

$$\underset{X \in \mathbb{P}}{\arg\min} \sum_{j=1}^{n} w_j \delta^2(X, A_j)$$

exists uniquely in $\mathbb{P}$. We call it the weighted Karcher mean (or the least squares mean), and denote as $\Lambda(\omega; \mathbb{A})$.

Motivated by the beginning work of J. Holbrook in $(\mathbb{P}, \delta)$, Y. Lim and M. Palfia have shown no dice theorem in a general Hadamard space. For our purpose, we explain it on the open convex cone $(\mathbb{P}, \delta)$.

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The Karcher means

For a positive probability vector $\omega = (w_1, w_2, \ldots, w_n)$, we denote

$$\overline{\omega} := (w_1, \ldots, w_n, w_1, \ldots, w_n, \ldots),$$

and $s(N) := \sum_{i=1}^{N} \overline{\omega}_i$ for each $N \in \mathbb{N}$, where $\overline{\omega}_i$ is the ith component of the infinite-dimensional vector $\overline{\omega}$.

The sequence of weighted inductive means is defined by

$$S_1(\omega; \mathbb{A}) = A_1, \; S_N(\omega; \mathbb{A}) = A_k \#_{\frac{s(N-1)}{s(N)}} S_{N-1}(\omega; \mathbb{A}), \; N \geqslant 2, \tag{5}$$

where $k \in \{1, \ldots, n\}$ is chosen so that $k \equiv N \pmod{n}$. Then

$$\lim_{N \to \infty} S_N(\omega; \mathbb{A}) = \Lambda(\omega; \mathbb{A}).$$

The Karcher means

Let $\mathbb{A} = (A, B, C) \in \mathbb{P}^3$ and $\omega = (1/3, 1/3, 1/3)$. Then

$$\begin{aligned}
S_1 &= A \\
S_2 &= B\#_{1/2}A \\
S_3 &= C\#_{2/3}(B\#_{1/2}A) = G_\omega(A, B, C) \\
S_4 &= A\#_{3/4}[C\#_{2/3}(B\#_{1/2}A)] \\
S_5 &= B\#_{4/5}\{A\#_{3/4}[C\#_{2/3}(B\#_{1/2}A)]\} \\
&\ \vdots \\
&\downarrow \\
&\Lambda(\omega; A, B, C).
\end{aligned}$$

The tensor product

The tensor product (or the Kronecker product) $A \otimes B$ of $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{s,t}$ is the $ms \times nt$ matrix given by

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

(1) $(A \otimes B)(C \otimes D) = AC \otimes BD$.

(2) For any invertible matrices A and B

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

(3) For positive definite matrices A, B and any real number t

$$(A \otimes B)^{t} = A^{t} \otimes B^{t}.$$

(4) The function $(A, B) \mapsto A \otimes B$ is continuous.

The tensor product

Let $A, B, C, D \in \mathbb{P}$. Then by the Riccati Lemma

$$(A\#B) \otimes (C\#D) = (A \otimes C)\#(B \otimes D).$$

Moreover, for $t \in [0, 1]$

$$(A\#_t B) \otimes (C\#_t D) = (A \otimes C)\#_t(B \otimes D).$$

It can be also generalized to the inductive mean. That is, for $\mathbb{A} = (A_1, \ldots, A_n), \mathbb{B} = (B_1, \ldots, B_n) \in \mathbb{P}^n$, and for a positive probability vector ω

$$S_N(\omega; \mathbb{A}) \otimes S_N(\omega; \mathbb{B}) = S_N(\omega; \mathbb{A} \otimes \mathbb{B}),$$

where $\mathbb{A} \otimes \mathbb{B} = (A_1 \otimes B_1, \ldots, A_n \otimes B_n) \in \mathbb{P}^n$.

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The tensor product

By no dice theorem and the continuity of tensor product,

$$\begin{aligned}
\Lambda(\omega; \mathbb{A} \otimes \mathbb{B}) &= \lim_{N\to\infty} S_N(\omega; \mathbb{A} \otimes \mathbb{B}) \\
&= \lim_{N\to\infty} [S_N(\omega; \mathbb{A}) \otimes S_N(\omega; \mathbb{B})] \\
&= \left[\lim_{N\to\infty} S_N(\omega; \mathbb{A})\right] \otimes \left[\lim_{N\to\infty} S_N(\omega; \mathbb{B})\right] \\
&= \Lambda(\omega; \mathbb{A}) \otimes \Lambda(\omega; \mathbb{B}).
\end{aligned}$$

The Hadamard product

For $A = [a_{ij}],\ B = [b_{ij}] \in M_{m,n}$, the Hadamard product (or the Schur product) $A \circ B$ is the $m \times n$ matrix of entry-wise products given by

$$A \circ B := [a_{ij} b_{ij}].$$

(1) The Hadamard product is a principal submatrix of the tensor product.

(2) There is a positive linear map Φ such that

$$\Phi(A_1 \otimes \cdots \otimes A_m) = A_1 \circ \cdots \circ A_m \tag{6}$$

for all $n \times n$ matrices $A_1, \ldots, A_m$.

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The Hadamard product of Karcher means

We simply denote as Γ_n the collection of all $n-$tuples $(\sigma_1, \sigma_2, \dots, \sigma_n)$ of permutations on n letters satisfying

$$\{\sigma_1(j), \sigma_2(j), \dots, \sigma_n(j)\} = \{1, 2, \dots, n\} \tag{7}$$

for each $j = 1, 2, \dots, n$.

Theorem

Let $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{P}^n$ and let $\omega = (w_1, \dots, w_n)$ be a positive probability vector. For any n-tuple of permutations $(\sigma_1, \dots, \sigma_n) \in \Gamma_n$,

$$(A_1^{-1} \circ \cdots \circ A_n^{-1})^{-1} \leqslant \Lambda(\omega; \mathbb{A}_{\sigma_1}) \circ \Lambda(\omega; \mathbb{A}_{\sigma_2}) \circ \cdots \circ \Lambda(\omega; \mathbb{A}_{\sigma_n}) \leqslant A_1 \circ \cdots \circ A_n.$$

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There is a positive linear map Φ satisfying

$$\Phi(\Lambda(\omega; \mathbb{A}_{\sigma_1}) \otimes \Lambda(\omega; \mathbb{A}_{\sigma_2}) \otimes \cdots \otimes \Lambda(\omega; \mathbb{A}_{\sigma_n})) = \Lambda(\omega; \mathbb{A}_{\sigma_1}) \circ \Lambda(\omega; \mathbb{A}_{\sigma_2}) \circ \cdots \circ \Lambda(\omega; \mathbb{A}_{\sigma_n}).$$

Then

$$\begin{aligned} & \Lambda(\omega; \mathbb{A}_{\sigma_1}) \circ \Lambda(\omega; \mathbb{A}_{\sigma_2}) \circ \cdots \circ \Lambda(\omega; \mathbb{A}_{\sigma_n}) \\ = \; & \Phi(\Lambda(\omega; \mathbb{A}_{\sigma_1}) \otimes \Lambda(\omega; \mathbb{A}_{\sigma_2}) \otimes \cdots \otimes \Lambda(\omega; \mathbb{A}_{\sigma_n})) \\ = \; & \Phi(\Lambda(\omega; \mathbb{A}_{\sigma_1} \otimes \mathbb{A}_{\sigma_2} \otimes \cdots \otimes \mathbb{A}_{\sigma_n})) \\ \leqslant \; & \Phi\left(\sum_{j=1}^{n} w_j A_{\sigma_1(j)} \otimes A_{\sigma_2(j)} \otimes \cdots \otimes A_{\sigma_n(j)} \right) \end{aligned}$$

The Hadamard product of Karcher means

$$
\begin{aligned}
& \Lambda(\omega; \mathbb{A}_{\sigma_1}) \circ \Lambda(\omega; \mathbb{A}_{\sigma_2}) \circ \cdots \circ \Lambda(\omega; \mathbb{A}_{\sigma_n}) \\
\leqslant\ & \Phi\left(\sum_{j=1}^{n} w_j A_{\sigma_1(j)} \otimes A_{\sigma_2(j)} \otimes \cdots \otimes A_{\sigma_n(j)}\right) \\
=\ & \sum_{j=1}^{n} w_j \Phi(A_{\sigma_1(j)} \otimes A_{\sigma_2(j)} \otimes \cdots \otimes A_{\sigma_n(j)}) \\
=\ & \sum_{j=1}^{n} w_j A_{\sigma_1(j)} \circ A_{\sigma_2(j)} \circ \cdots \circ A_{\sigma_n(j)} \\
=\ & \sum_{j=1}^{n} w_j A_1 \circ A_2 \circ \cdots \circ A_n \\
=\ & A_1 \circ A_2 \circ \cdots \circ A_n.
\end{aligned}
$$

The Hadamard product of Karcher means

Using Choi's inequality and the self-duality of weighted Karcher means,

$$
\begin{aligned}
&\left[A_1^{-1} \circ \cdots \circ A_n^{-1}\right]^{-1} \\
&\leqslant \left[\Lambda(\omega; \mathbb{A}_{\sigma_1}^{-1}) \circ \Lambda(\omega; \mathbb{A}_{\sigma_2}^{-1}) \circ \cdots \circ \Lambda(\omega; \mathbb{A}_{\sigma_n}^{-1})\right]^{-1} \\
&= \left[\Phi\left(\Lambda(\omega; \mathbb{A}_{\sigma_1}^{-1}) \otimes \Lambda(\omega; \mathbb{A}_{\sigma_2}^{-1}) \otimes \cdots \otimes \Lambda(\omega; \mathbb{A}_{\sigma_n}^{-1})\right)\right]^{-1} \\
&\leqslant \Phi\left(\left[\Lambda(\omega; \mathbb{A}_{\sigma_1}^{-1}) \otimes \Lambda(\omega; \mathbb{A}_{\sigma_2}^{-1}) \otimes \cdots \otimes \Lambda(\omega; \mathbb{A}_{\sigma_n}^{-1})\right]^{-1}\right) \\
&= \Phi\left(\Lambda(\omega; \mathbb{A}_{\sigma_1}^{-1})^{-1} \otimes \Lambda(\omega; \mathbb{A}_{\sigma_2}^{-1})^{-1} \otimes \cdots \otimes \Lambda(\omega; \mathbb{A}_{\sigma_n}^{-1})^{-1}\right) \\
&= \Phi\left(\Lambda(\omega; \mathbb{A}_{\sigma_1}) \otimes \Lambda(\omega; \mathbb{A}_{\sigma_2}) \otimes \cdots \otimes \Lambda(\omega; \mathbb{A}_{\sigma_n})\right) \\
&= \Lambda(\omega; \mathbb{A}_{\sigma_1}) \circ \Lambda(\omega; \mathbb{A}_{\sigma_2}) \circ \cdots \circ \Lambda(\omega; \mathbb{A}_{\sigma_n}).
\end{aligned}
$$

H. Lee and S. Kim, *The Hadamard product for the weighted Karcher means*, Linear Algebra and its Applications, **501** (2016), 290-303.

Open problems

Let $\omega = (w_1, \ldots, w_n), \phi = (\mu_1, \ldots, \mu_n)$ be probability vectors with $\omega \prec \phi$:

$$\sum_{i=1}^{k} \mu_i^{\downarrow} \geqslant \sum_{i=1}^{k} w_i^{\downarrow} \quad \text{for} \quad k = 1, 2, \ldots, n-1,$$

and the equality holds for $k = n$, where $(w_1^{\downarrow}, \ldots, w_n^{\downarrow})$ is the vector with the same components of ω, but sorted in descending order.

Then does the following hold?

$$\begin{aligned} &\Lambda(\omega; \mathbb{A}_{\sigma_1}) \circ \Lambda(\omega; \mathbb{A}_{\sigma_2}) \circ \cdots \circ \Lambda(\omega; \mathbb{A}_{\sigma_n}) \\ &\qquad \leqslant \Lambda(\phi; \mathbb{A}_{\sigma_1}) \circ \Lambda(\phi; \mathbb{A}_{\sigma_2}) \circ \cdots \circ \Lambda(\phi; \mathbb{A}_{\sigma_n}) \end{aligned} \tag{8}$$

for any n-tuple of permutations $(\sigma_1, \ldots, \sigma_n) \in \Gamma_n$.

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Thank You for Your Attention!