Characterization of centralizers and Jordan centralizers on non-self-adjoint operator algebras by local action

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Contents

Introduction

Let $\mathcal{A}$ be an associative algebra over a field $\mathbb{F}$, and $\Phi : \mathcal{A} \to \mathcal{A}$ be a linear mapping. Φ is a left centralizer or multiplier if

$$\Phi(AB) = \Phi(A)B, \qquad \forall A, B \in \mathcal{A}.$$

Φ is a right centralizer or multiplier if

$$\Phi(AB) = A\Phi(B), \qquad \forall A, B \in \mathcal{A}.$$

Φ is called a centralizer if it is both left and right centralizer[1-4].

Introduction

Φ is a left Jordan centralizer if

$$\Phi(A^2) = \Phi(A)A, \qquad \forall A \in \mathcal{A},$$

equivalently,

$$\Phi(AB + BA) = \Phi(A)B + \Phi(B)A, \qquad \forall A, B \in \mathcal{A}.$$

Φ is a right Jordan centralizer if

$$\Phi(A^2) = A\Phi(A), \qquad \forall A \in \mathcal{A},$$

equivalently,

$$\Phi(AB + BA) = A\Phi(B) + B\Phi(A), \qquad \forall A, B \in \mathcal{A}.$$

Φ is called a Jordan centralizer if it is both left and right Jordan centralizer[5].

Introduction

In this talk, we mainly discuss centralizers and Jordan centralizers on non-self-adjoint operator algebras by local action.

In general there are two directions in the study of the local actions of mappings of operator algebras. One is the well known local mappings problem, such as local derivations and local automorphisms. The other direction is to study conditions under which mappings of operator algebras can be completely determined by the action on some sets of operators.

Introduction

local actions

We call a linear mapping Φ a local left (right) centralizer of $\mathcal{A}$, if for each $A \in \mathcal{A}$ there is a left (right) centralizer Φ_A of $\mathcal{A}$ such that $\Phi(A) = \Phi_A(A)$.

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local actions

Φ is called local centralizer if it is both local left centralizer and local right centralizer.

Introduction

Wei and Li 2007 [6]

Let $\mathcal{L}$ be a $\mathcal{J}-$ subspace lattice on a Banach space X and $\mathcal{A}$ be a standard subalgebra of $Alg\mathcal{L}$. Then every linear local left (right) centralizer of $\mathcal{A}$ is necessarily a left (right) centralizer.

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Questions

How to characterize local centralizer on nest subalgebras of factor von Neumann algebras?

Introduction

The nest subalgebras of von Neumann algebras was first studied by Gilfeather and Larson [7]. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H. A nest β in $\mathcal{M}$ is a totally ordered family of projections in $\mathcal{M}$ which is closed in the strong operator topology, and which includes 0 and I. A nest is said to be non-trivial if it contains at least one non-trivial projection. The nest subalgebra of $\mathcal{M}$ associated to a nest β is the set

$$\mathrm{alg}_{\mathcal{M}}\beta = \{X \in \mathcal{M} : PXP = XP, \quad \forall P \in \beta\}.$$

Introduction

local actions

A linear mapping $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be left (right) centralized at a given point $G \in \mathcal{A}$ if $\Phi(AB) = \Phi(A)B$ $(\Phi(AB) = A\Phi(B))$ for all A, $B \in \mathcal{A}$ with $AB = G$.

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local actions

Φ is called centralized at G if it is both left centralized and right centralized at G.

Introduction

Brešer 2007 [8]

If $\mathcal{A}$ is a prime ring containing a nontrivial idempotent and Φ is right centralized at zero point, then Φ is a right centralizer on $\mathcal{A}$.

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Qi and Hou 2013 [9]

Let $\mathcal{U}$ be the triangular ring and $\Phi : \mathcal{U} \to \mathcal{U}$ is an additive map. Then Φ is centralized at zero point if and only if Φ is a centralizer.

Introduction

Definition

We say a linear mapping $\Phi : \mathcal{A} \to \mathcal{A}$ is left Jordan centralized at a given point $G \in \mathcal{A}$ if

$$\Phi(AB + BA) = \Phi(A)B + \Phi(B)A$$

holds for all A, $B \in \mathcal{A}$ with $AB = G$.

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Definition

Φ is right Jordan centralized at a given point $G \in \mathcal{A}$ if

$$\Phi(AB + BA) = A\Phi(B) + B\Phi(A)$$

holds for all A, $B \in \mathcal{A}$ with $AB = G$.

Introduction

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Questions

How to characterize the mappings which are Jordan centralized at a general given point?

Introduction

The triangular algebras were firstly introduced in [10]. Let $\mathcal{A}$ and $\mathcal{C}$ be two algebras over a field $\mathbb{F}$ with unit I_1 and I_2, respectively, and let $\mathcal{M}$ be a faithful ($\mathcal{A}$, $\mathcal{C}$)-bimodule, that is, $\mathcal{M}$ is a ($\mathcal{A}$, $\mathcal{C}$)-bimodule satisfying, for $X \in \mathcal{A}$, if $X\mathcal{M} = \{0\}$, then $X = 0$ (i.e. $\mathcal{M}$ is a faithful left $\mathcal{A}$-module), and for $Z \in \mathcal{C}$, if $\mathcal{M}Z = \{0\}$, then $Z = 0$ (i.e. $\mathcal{M}$ is a faithful right $\mathcal{C}$-module).

Recall that the algebra

$$\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{C}) = \{\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} : X \in \mathcal{A},\ Y \in \mathcal{M},\ Z \in \mathcal{C}\},$$

under the usual matrix addition and formal matrix multiplication will be called a triangular algebra.

Main results

We give a characterization of local centralizers on nest subalgebras of factor von Neumann algebras.

Theorem 1

Let β be a non-trivial nest in an arbitrary factor von Neumann algebra $\mathcal{M}$, and Φ be a norm continuous linear local left (right) centralizer from the nest subalgebra $\mathrm{alg}_{\mathcal{M}}\beta$ into $\mathcal{M}$. Then Φ is a left (right) centralizer.

Main results

We give a characterization of left Jordan centralized mappings at a given point.

Theorem 2

Let $\mathcal{A}$ and $\mathcal{C}$ be two algebras over a number field $\mathbb{F}$ with unit I_1 and I_2, respectively, and $\mathcal{M}$ be a faithful left $\mathcal{A}$-module. The triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{C})$ is written for $\mathcal{T}$. Suppose that

(i) For every $X \in \mathcal{A}$, there is some integer n such that $nI_1 - X$ is invertible,

(ii) For every $Z \in \mathcal{C}$, there is some integer n such that $nI_2 - Z$ is invertible.

Main results

Theorem 2

If $\phi : \mathcal{T} \to \mathcal{T}$ is a left Jordan centralized mapping at a given point $G = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in \mathcal{T}$, then there exists an element $D \in \mathcal{A}$, two linear mappings $h_{12} : \mathcal{C} \to \mathcal{M}$ satisfying $h_{12}(ZW + WZ) = h_{12}(Z)W + h_{12}(W)Z$, and $h_{22} : \mathcal{C} \to \mathcal{C}$ satisfying $h_{22}(ZW + WZ) = h_{22}(Z)W + h_{22}(W)Z$ for all $Z, W \in \mathcal{C}$ with $ZW = C$ such that

$$\phi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} DX & DY + h_{12}(Z) \\ 0 & h_{22}(Z) \end{bmatrix}, \quad \forall \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathcal{T}.$$

Main results

Similarly, we give a characterization of right Jordan centralized mappings at a given point.

Theorem 3

Let $\mathcal{A}$ and $\mathcal{C}$ be two algebras over a number field $\mathbb{F}$ with unit I_1 and I_2, respectively, and $\mathcal{M}$ be a faithful right $\mathcal{C}$-module. The triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{C})$ is written for $\mathcal{T}$. Suppose that

(i) For every $X \in \mathcal{A}$, there is some integer n such that $nI_1 - X$ is invertible,

(ii) For every $Z \in \mathcal{C}$, there is some integer n such that $nI_2 - Z$ is invertible.

Main results

Theorem 3

If $\phi : \mathcal{T} \to \mathcal{T}$ is a right Jordan centralized mapping at a given point $G = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in \mathcal{T}$, then there exists an element $E \in \mathcal{C}$, two linear mappings $f_{11} : \mathcal{A} \to \mathcal{A}$ satisfying $f_{11}(XU + UX) = Xf_{11}(U) + Uf_{11}(X)$, and $f_{12} : \mathcal{A} \to \mathcal{M}$ satisfying $f_{12}(XU + UX) = Xf_{12}(U) + Uf_{12}(X)$ for all $X, U \in \mathcal{A}$ with $XU = A$ such that

$$\phi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} f_{11}(X) & YE + f_{12}(X) \\ 0 & ZE \end{bmatrix}, \ \forall \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathcal{T}.$$

Main results

The following result states that the Jordan centralizers on triangular algebras can be determined by their action on a given point.

Theorem 4

Let $\mathcal{A}$ and $\mathcal{C}$ be two algebras over a number field $\mathbb{F}$ with unit I_1 and I_2, respectively, and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{C})$-bimodule. The triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{C})$ is written for $\mathcal{T}$. Suppose that

(i) For every $X \in \mathcal{A}$, there is some integer n such that $nI_1 - X$ is invertible,

(ii) For every $Z \in \mathcal{C}$, there is some integer n such that $nI_2 - Z$ is invertible.

If $\phi : \mathcal{T} \to \mathcal{T}$ is a Jordan centralized mapping at a given point $G \in \mathcal{T}$, then ϕ is a centralizer.

Application

We apply Theorem 4 to some non-self-adjoint operator algebras. For irreducible CDC algebras and Banach space nest algebras, we have the following Theorems.

Theorem 5

Let $alg\mathcal{L}$ be an irreducible CDC algebra on a complex Hilbert space H. Then every linear mapping from $alg\mathcal{L}$ into itself Jordan centralized at a given point $G \in alg\mathcal{L}$ is a centralizer.

Application

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Theorem 5

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Theorem 6

Let $\mathcal{N}$ be a nest on a complex Banach space space. Suppose that there exists a non-trivial element in $\mathcal{N}$ which is complemented in X. If $\phi : alg\mathcal{N} \to alg\mathcal{N}$ is a Jordan centralized mapping at a given point $G \in alg\mathcal{N}$, then ϕ is a centralizer.

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Thank you for your attention!