Algebraic curves associated to some matrices

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Outline

The complex projective algebraic curve C_F of an arbitrary ternary homogeneous polynomial $F(t, x, y)$ is defined by

$$C_F = \{[(t, x, y)] \in \mathbf{CP}^2 : F(t, x, y) = 0\}.$$

where $\mathbf{CP}^2$ denotes the complex projective plane, that is, the quotient space $\mathbb{C}^3 \setminus \{(0, 0, 0)\}/_\sim$ with respect to the relation $(t_1, x_1, y_1) \sim (t_2, x_2, y_2)$ if $(t_2, x_2, y_2) = c(t_1, x_1, y_1)$ for a non-zero complex number c.

The dual curve of C_F is defined by

$$\Gamma_F = \{[(T_0, X_0, Y_0)] \in \mathbf{CP}^2 : T_0 t + X_0 x + Y_0 y = 0 \text{ is a tangent line of } C_F\}.$$

1. Introduction

Let A be an $n \times n$ matrix.

Denote $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/(2i)$.

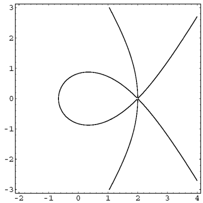
A complex projective algebraic curve associated to A is defined by the associated homogeneous polynomial

$$F_A(t, x, y) = \det\Big(tI_n + x\,\Re(A) + y\,\Im(A)\Big).$$

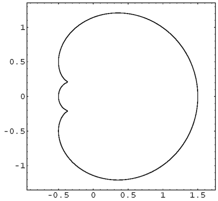
Numerical range

$$W(A) = \{x^* A x; x \in \mathbf{C}^n, \|x\| = 1\}.$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$



$F_A(1, x, y) = 0$



Dual curve (flat portion)

1. Introduction

For arbitrary complex numbers $a_1, a_2, \dots, a_n$, we consider an $n \times n$ cyclic weighted shift matrix $S(a_1, a_2, \dots, a_n)$ defined as

$$S = S(a_1, a_2, \dots, a_n) = \begin{pmatrix} 0 & a_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & a_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \vdots & \vdots & \ddots & a_{n-1} \\ a_n & 0 & \dots & \dots & \dots & 0 \end{pmatrix}.$$

1. Introduction

- M. C. Tsai, P. Y. Wu, 2011
  M. C. Tsai, H. L. Gau, H. C. Wang, 2014

  Let S be an $n \times n$ cyclic weighted shift matrix. Then the boundary of $W(S)$ has a flat portion if and only if the weights are nonzero and the numerical ranges of three $(n-1) \times (n-1)$ principal submatrices of S are equal.

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- H. L. Gau, M. C. Tsai, H. C. Wang, 2013

  A cyclic weighted shift matrix $S(a_1, a_2, \ldots, a_n)$ is unitarily irreducible if and only if its weights $a_1, a_2, \ldots, a_n$ are non-periodic.

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1. Investigate the types of the singular points of the curve $F_S(t, x, y) = 0$.
2. Find the number of the singular points of $F_S(t, x, y) = 0$.
3. Extend to general hyperbolic forms.

2. Singular points

A point (t_0, x_0, y_0) of the curve $F(t, x, y) = 0$ is called a singular point if $F(t_0, x_0, y_0) = 0$ and

$$\frac{\partial}{\partial t}F(t_0, x_0, y_0) = \frac{\partial}{\partial x}F(t_0, x_0, y_0) = \frac{\partial}{\partial y}F(t_0, x_0, y_0) = 0.$$

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- If the boundary of $W(A)$ has a flat portion on the line $a_0 + a_1\, x + a_2\, y = 0$ then the point (a_0, a_1, a_2) is a singular point of the curve $F_A(t, x, y) = 0$.

2. Singular points

A singular point (t_0, x_0, y_0) of the curve is called a double point if at least one of the second derivatives

$$\frac{\partial^2}{\partial t^2}F(t_0, x_0, y_0),\ \frac{\partial^2}{\partial x^2}F(t_0, x_0, y_0),\ \frac{\partial^2}{\partial y^2}F(t_0, x_0, y_0),$$

$$\frac{\partial^2}{\partial t \partial x}F(t_0, x_0, y_0),\ \frac{\partial^2}{\partial t \partial y}F(t_0, x_0, y_0),\ \frac{\partial^2}{\partial x \partial y}F(t_0, x_0, y_0)$$

does not vanish.

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does not vanish.

A double point of a curve is called a node(or an ordinary double point) if there are two distinct tangents at the point.

2. Singular points

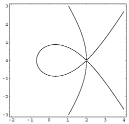
Nakazato-Chien, 2012

The shapes of the numerical ranges of 4×4 matrices A are determined according to the classification of singular points of $F_A(t, x, y) = 0$

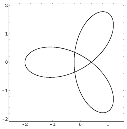
For an irreducible ternary form, there are 21 types according to the number of its singular points and their forms

- 1 triple point
- 3 nodes
- 2 nodes
- 1 node
- 1 node and 1 tacnode
- 1 osnode
- 1 tacnode
- 2 cusps
- 2 cusps and 1 node
- no singular points

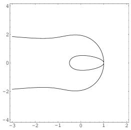
2. Singular points



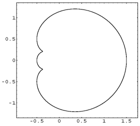
triple point



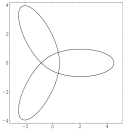
3 nodes



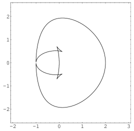
tacnode



one flat



3 flats



no flat

2. Singular points

Nakazato-Chien, 2013

Theorem 2.1 Let $S(a_1, a_2, \ldots, a_n)$ be a cyclic weighted shift matrix with non-zero weights. Then all singular points of the complex projective curve $F_S(t, x, y) = 0$ are nodes.

2. Singular points

Nakazato-Chien, 2013

Theorem 2.1 Let $S(a_1, a_2, \ldots, a_n)$ be a cyclic weighted shift matrix with non-zero weights. Then all singular points of the complex projective curve $F_S(t, x, y) = 0$ are nodes.

- The singular points of the curve $F_S(t, x, y) = 0$ belong to the real projective plane.
- A real singular point of $F_S(t, x, y) = 0$ on the line $t = 0$ can be assumed $(0, 1, 0)$, the tangents are $t = \pm ay$.
- A real singular point $(1, x_0, y_0)$ of $F_S(t, x, y) = 0$ can be assumed $(1, 1, 0)$, the tangents are $\pm by + x - 1 = 0$.

Nakazato-Chien, 2013

Theorem 2.2 Let $S(a_1, a_2, \ldots, a_n)$ be a cyclic weighted shift matrix with non-zero weights. Then the number of singular points of the curve $F_S(t, x, y) = 0$ is at most $n(n-1)/2$.

2. Singular points

Nakazato-Chien, 2013

Theorem 2.2 Let $S(a_1, a_2, \ldots, a_n)$ be a cyclic weighted shift matrix with non-zero weights. Then the number of singular points of the curve $F_S(t, x, y) = 0$ is at most $n(n-1)/2$.

- $F_S(t, x, y)$ has no repeated factors.
- Then, by an algebraic curve theorem,
  the number of double points $\leq n(n-1)/2$.

How many singular points of $F_S(t, x, y) = 0$?

① $n(n-1)/2$

The upper bound $n(n-1)/2$ is attained for any $n \geq 3$ by the canonical cyclic shift matrix $S(1, 1, \ldots, 1)$.

$$F_S(t, x, y) = \prod_{k=0}^{n-1} \Big(t + \cos(2k\pi/n)x + \sin(2k\pi/n)y \Big).$$

The singular points of the curve $F_S(t, x, y) = 0$ are given by

$$\{(t, x, y) = (1, -\frac{\cos((k+\ell)\pi/n)}{\cos((k-\ell)\pi/n)}, -\frac{\sin((k+\ell)\pi/n)}{\cos((k-\ell)\pi/n)}) : 0 \leq \ell < k \leq n\}.$$

① $n(n-1)/2$ ② $n(n-2)/2$

Nakazato-Chien, 2013

Theorem 2.3 Let $n \geq 4$ be an even number and $S(a_1, a_2, \ldots, a_n)$ be a cyclic weighted shift matrix with non-zero weights. Then the number of singular points of the curve $F_S(t, x, y) = 0$ is $n(n-2)/2$ if and only if the form $F_S(t, x, y)$ is the product of $n/2$ quadratic forms.
In this case, $S(a_1, a_2, \ldots, a_n)$ is unitarily reducible and the weights are 2-periodic, that is, $a_{2j-1} = \alpha, a_{2j} = \beta$, $j = 1, 2, \ldots, n/2$.

2. Singular points

① $n(n-1)/2$ ② $n(n-2)/2$ ③ $n(n-3)/2$

Nakazato-Chien, 2013

Theorem 2.4 Let $S = S(a_1, a_2, \ldots, a_n)$, $n \geq 3$, be a unitarily irreducible cyclic weighted shift matrix with non-zero weights. Then the number of singular points of the associative curve $F_S(t, x, y) = 0$ is at most $n(n-3)/2$.

① $n(n-1)/2$ ② $n(n-2)/2$ ③ $n(n-3)/2$

Nakazato-Chien, 2013

Theorem 2.4 Let $S = S(a_1, a_2, \dots, a_n)$, $n \geq 3$, be a unitarily irreducible cyclic weighted shift matrix with non-zero weights. Then the number of singular points of the associative curve $F_S(t, x, y) = 0$ is at most $n(n-3)/2$.

- Suppose $(1, x_0, y_0)$ is a singular point of $F_S(t, x, y) = 0$. By the symmetry,

  $$F_S(t, \cos(2\pi/n)x - \sin(2\pi/n)y, \sin(2\pi/n)x + \cos(2\pi/n)y) = F_S(t, x, y),$$

  the n points

  $$(1, x_0 \cos(2k\pi/n) - y_0 \sin(2k\pi/n), x_0 \sin(2k\pi/n) + y_0 \cos(2k\pi/n))$$

  $k = 0, 1, 2, \dots, n-1$ are singular points of $F_S(t, x, y) = 0$.

2. Singular points

① $n(n-1)/2$ ② $n(n-2)/2$ ③ $n(n-3)/2$

- n is an odd number: Then the curve $F_S(t,x,y)=0$ has no singular points on the line $t=0$. Hence the possible numbers of singular points of $F_S(t,x,y)=0$ are

$$\{n(n-1)/2, n(n-3)/2, n(n-5)/2, \dots, n, 0\}.$$

- n is an even number ≥ 4 : If $(0,x_0,y_0)$ is a singular point of the curve $F_S(t,x,y)=0$, where $(x_0,y_0)\in\mathbb{R}^2$, then the $n/2$ points

$$(0, x_0\cos(2k\pi/n) - y_0\sin(2k\pi/n), x_0\sin(2k\pi/n) + y_0\cos(2k\pi/n))$$

$k=0,1,2,\dots,(n/2)-1$, are singular points of the curve $F_S(t,x,y)=0$. Hence, the number of singular points of the curve $F_S(t,x,y)=0$ is one of the following:

$$\{n(n-1)/2, n(n-2)/2, n(n-3)/2, \dots, n, n/2, 0\}.$$

Conjecture

The upper bound $n(n-3)/2$ is sharp for the numbers of the singular points of $F_S(t,x,y)=0$ associated with $n \times n$ unitarily irreducible cyclic weighted shift matrices.

Partial Answer: The conjecture is true for $n = 4, 5, 6, 7$.

For examples:

$n = 4$. $a_1 a_3 = a_2 a_4$ and $(a_1 - a_3)^2 + (a_2 - a_4)^2 > 0$, the curve $F_S(t, x, y) = 0$ has $n(n-3)/2 = 2$ nodes at $(t, x, y) = (0, 1, 0), (0, 0, 1)$.

Partial Answer: The conjecture is true for $n = 4, 5, 6, 7$.

For examples:

$n = 4$. $a_1 a_3 = a_2 a_4$ and $(a_1 - a_3)^2 + (a_2 - a_4)^2 > 0$, the curve $F_S(t, x, y) = 0$ has $n(n-3)/2 = 2$ nodes at $(t, x, y) = (0, 1, 0), (0, 0, 1)$.

$n = 6$. $a_1 = a_6 = 2\sqrt{2}, a_2 = a_5 = 2$ and $a_3 = a_4 = \sqrt{8/3}$, the curve $F_S(t, x, y) = 0$ has $n(n-3)/2 = 9$ nodes at $(t, x, y) = (1, \cos(k\pi/3), \sin(k\pi/3))$, $k = 0, 1, \ldots, 5$, and $(t, x, y) = (0, 0, 1), (0, \cos(\pi/6), \sin(\pi/6)), (0, \cos(\pi/6), \sin(\pi/6))$.

The real ternary form $F_S(t,x,y)$ of a cyclic weighted shift matrix $S = S(a_1, a_2, \ldots, a_n)$ with non-zero weights satisfies the following conditions:

(*i*) $F_S(t,x,y)$ is hyperbolic w.r.t. $(1,0,0)$ and $F(1,0,0) = 1$.

(*ii*) $F_S(t,x,y)$ is weakly circular symmetric:

$$F_S(t, \cos(2\pi/n)\,x - \sin(2\pi/n)\,y, \sin(2\pi/n)\,x + \cos(2\pi/n)\,y) = F_S(t,x,y).$$

(*iii*) $F_S(t,x,-y) = F_S(t,x,y)$.

(*iv*) $F_S(t,-1,-i) = t^n - a$ for some nonzero real number a.

Theorem 3.1 Let $F(t, x, y)$ be a real ternary form of degree n satisfying conditions (i)-(iv). Then all singular points of the complex projective curve $F(t, x, y) = 0$ are nodes.

3. General hyperbolic forms

Theorem 3.1 Let $F(t, x, y)$ be a real ternary form of degree n satisfying conditions (i)-(iv). Then all singular points of the complex projective curve $F(t, x, y) = 0$ are nodes.

Theorem 3.2 Let $F(t, x, y)$ be a real ternary form of degree n satisfying conditions (i)-(iv). Then $F(t, x, y)$ has no repeated factors, and the number of singular points of the complex projective curve $F(t, x, y) = 0$ is at most $n(n-1)/2$.

References

[1] Mao-Ting Chien and Hiroshi Nakazato, Hyperbolic forms associated with cyclic weighted shift matrices, Linear Algebra and Its Applications, 439(2013), 3541-3554.

[2] Mao-Ting Chien and Hiroshi Nakazato, Singular points of cyclic weighted shift matrices, Linear Algebra and Its Applications, 439(2013), 4090-4100.

[3] Mao-Ting Chien and Hiroshi Nakazato, Singular points of the algebraic curves of symmetric hyperbolic forms, Linear Algebra and Its Applications, 470(2015), 40-50.

Thank you for your attention!