# Sofic groups

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- The **Gottschalk's surjunctivity conjecture** states that for every countable discrete group $G$, every injective continuous function $f : A^G \to A^G$ satisfying $f(g \cdot x) = g \cdot f(x)$ for every $g \in G, x \in A^G$ is surjective.

# Kaplansky's direct finiteness conjecture

- Let $R$ be a unit ring. We denote by $RG$ the group ring with coefficients in $R$. It consists of finitely supported $R$-valued functions $f$ on $G$, which we write $\sum_{s\in G} f_s s$. The algebraic structure is defined by $(\sum_{s\in G} f_s s) + (\sum_{s\in G} g_s s) = \sum_{s\in G}(f_s + g_s)s$ and $(\sum_{s\in G} f_s s)(\sum_{s\in G} g_s s) = \sum_{s\in G}(\sum_{t\in G} f_t g_{t^{-1}s})s$.

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- A group $G$ satisfies Kaplansky's direct finiteness conjecture if and only if $KG$ is directly finite for every finite field $K$.
- Gottschalk's surjunctivity conjecture is stronger than Kaplansky's direct finiteness conjecture.

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- Defining

$$\Big(\sum_g a_g g\Big) \cdot \Big(\sum_g b_g g\Big) = \sum_g \Big( \sum_{hh'=g} a_h b_{h'} \Big) g,$$

for every $\sum_g a_g g \in K^G$ and $\sum_g b_g g \in KG$ one obtains a right action of $KG$ on $K^G$ that extends the multiplication operation in $KG$ and commutes with the left action of $G$ on $KG$.

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- Define the continuous equivariant map $f : K^G \to K^G$ by $f(x) := x \cdot a$ then $f$ is injective.

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- Define the continuous equivariant map $f : K^G \to K^G$ by $f(x) := x \cdot a$ then $f$ is injective.
- Since $G$ satisfies Gottschalk's surjunctivity conjecture, the map $f$ is surjective.
- Hence there exists $b_0 \in K^G$ such that $b_0 \cdot a = 1$. As $a$ has a left inverse $b_0$ and has a right inverse $b$, we must have $b = b_0$.

# Sofic group

- For $d \in \mathbb{N}$, denote by $\mathrm{Sym}(d)$ the permutation group of $\{1, ..., d\}$. The Hamming metric on $\mathrm{Sym}(d)$:

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- Gromov (1999): $G$ is sofic if for any nonempty finite $F \subset G$ and $\varepsilon > 0$, there are some $d \in \mathbb{N}$ and $\sigma : G \to \mathrm{Sym}(d)$ with
  1) (Nearly homomorphism) $\rho(\sigma_{st}, \sigma_s \sigma_t) < \varepsilon$ for all $s, t \in F$, and
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  2) (Nearly injective) $\rho(\sigma_s, \sigma_t) > 1 - \varepsilon$ for all distinct $s, t \in F$.
- Equivalently, there exists a sequence of maps $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}_{i \in \mathbb{N}}$ such that
  1. For any $s, t \in G$,
  $$\lim_{i \to \infty} \rho(\sigma_{i,st}, \sigma_{i,s}\sigma_{i,t}) \to 0,$$
  2. For any distinct $s, t \in G$,
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- A group $G$ is residually finite if for every non-identity element $g \in G$, there exists a nornmal subgroup $N$ of $G$ with finite index such that $g \notin N$, i.e. there exist a finite group $F$ and a homomorphism $\varphi : G \to F$ such that $\varphi(g) \neq 1_F$.

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- Every permutation group $\mathrm{Sym}(n)$ canonically embeds into the unitary group $U(n)$ and their distances satisfy

$$d_{Hamm}(\sigma, \tau) = \frac{1}{2}(d_{HS}(A_\sigma, A_\tau))^2.$$

# Gottschalk's surjunctive conjecture

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- Gromov (1999): Gottschalk's surjunctivity conjecture is true for all sofic groups.

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- This argument also works for residually finite groups.
- Gromov (1999): Gottschalk's surjunctivity conjecture is true for all sofic groups.
- Corollary: Kaplansky's direct finiteness conjecture is true for all sofic groups.

- In a breakthrough result, Lewis Bowen introduced measure sofic measure entropy to classify Bernoulli shifts over sofic groups (2010).

# Entropy Theory of Sofic Groups (Beyond Amenability)

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- Lewis Bowen asked whether expansive systems of actions of sofic groups have measures with maximal entropy.
- Yes. We give a sufficient condition for the existence of measure of maximal sofic entropy:asymptotically h-expansiveness. As a consequence, we give a positive answer for Bowen's question. (C-Zhang 2015)

- **Variational Principle**: (Ruelle) For $G \curvearrowright X$, where $G = \mathbb{Z}^d$ for some $d \in \mathbb{N}$ and $f \in C(X)$,

$$P(f, X, G) = \sup_{\mu \in M_G(X)} \{h_\mu(X, G) + \int_X f d\mu\},$$

where $M_G(X)$ is the set of all $G$-invariant Borel probability measures on $X$ and $h_\mu(X, G)$ is the the Kolomogorov-Sinai measure entropy of the action with respect to $\mu$.

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- (C, 2013) I defined topological pressure and established the variational principle for actions of sofic groups.
  $G$ a countable sofic group, $G \curvearrowright X$, $f \in C(X)$, $\Sigma$ a sofic approximation sequence of $G$. Then

$$h_\Sigma(f, X, G) = \sup\left\{h_{\Sigma,\mu}(X, G) + \int_X f d\mu : \mu \in M_G(X)\right\}.$$

- Example (C, 2013): $G$: countable sofic group, $\Sigma$: a sofic approximation sequence of $G$ $k \in \mathbb{N}$ and $X = \{0, 1, ..., k-1\}^G$. Let $a_0, ..., a_{k-1} \in \mathbb{R}$ and define $f \in C(X)$ by $f(x) = a_{x_e}$ where $x = (x_t)_{t \in G}$. $G \curvearrowright X^G$ by the left shifts $s \cdot (x_t)_{t \in G} = (x_{s^{-1}t})_{t \in G}$. Then the toplogical pressure of $f$,

$$h_\Sigma(f, X, G) = \log\Big(\sum_{j=0}^{k-1} \exp(a_j)\Big).$$

Furthermore, the measure $\mu^G$ is an equilibrium state for $f$, where $\mu$ the probability measure on $\{0, ..., k-1\}$, defined by

$$\mu(i) = \frac{\exp(a_i)}{\sum_{j=0}^{k-1} \exp(a_j)}, \text{ for all } 0 \le i \le k-1.$$