Product of positive semi-definite matrices

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July 4, 2016

2016 Workshop on Matrices and Operators
Jeju National University, Korea

Based on a joint work with
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Research Grants Council 研究資助局

Outline

- Product of positive definite matrices
- Product of positive semi-definite matrices
- Our result
- Summary

Question

Question: Given $n \times n$ complex $A \in M_n$, when A can be wriitten as a product of some positive definite / semi-definite matriices, i.e.,

$$A = P_1 P_2 P_3 \cdots P_k,$$

where $P_1, \ldots, P_k$ are positive definite/semi-definite matrices?

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$$A = \begin{bmatrix} 3 & 21 \\ 1 & -1 \end{bmatrix}$$

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Example:

$$A = \begin{bmatrix} 3 & 21 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

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Can A be written as a product of two positive definite matrices?

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Can A be written as a product of two positive definite matrices?

Define

$$k = \min\{k : A = P_1 P_2 \cdots P_k \text{ with positive semi-definite } P_j\}.$$

How to determine k?

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Nonsingular matrix

- When A is nonsingular, then

$$\begin{aligned} A &= P_1 P_2 P_3 \cdots P_k \\ \Longrightarrow \quad \det(A) &= \det(P_1)\det(P_2)\det(P_3)\cdots\det(P_k) > 0 \end{aligned}$$

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- **Necessary condition:**
  An nonsingular matrix $A \in M_n$ can be written as the product of positive definite matrices only when $\det(A) > 0$.

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Necessary and sufficient condition [Ballantine, LAA 3:79-114 3(1970)]

- An nonsingular matrix $A \in M_n$ can be written as the product of positive definite matrices if and only if $\det(A) > 0$.
- In particular, it can be written as the product of at most five positive definite matrices.

Nonsingular matrix

Proposition

Suppose A is a product of k positive definite matrices and $S \in M_n$ is invertible. Then

1. If k is even, then $S^{-1}AS$ is a product of k positive definite matrices.
2. If k is odd, then S^*AS is a product of k positive definite matrices.

Nonsingular matrix

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1. If k is even, then $S^{-1}AS$ is a product of k positive definite matrices.
2. If k is odd, then S^*AS is a product of k positive definite matrices.

When $k = 2m$,

$$\begin{aligned} A &= P_1 P_2 P_3 \cdots P_{2m-1} P_{2m} \\ S^{-1}AS &= S^{-1}P_1(S^{-1})^* \, S^*P_2S \, S^{-1}P_3(S^{-1})^* \, \cdots \, S^{-1}P_{2m-1}(S^{-1})^* \, S^*P_{2m}S \end{aligned}$$

When $k = 2m+1$,

$$\begin{aligned} A &= P_1 P_2 P_3 \cdots P_{2m} P_{2m+1} \\ S^*AS &= S^*P_1S \, S^{-1}P_2(S^{-1})^* \, S^*P_3S \, \cdots \, S^{-1}P_{2m}(S^{-1})^* S^*P_{2m+1}S \end{aligned}$$

Nonsingular matrix

At most four positive definite matrices [Ballantine, LAA 3:79-114 3(1970)]

Suppose A is non-scalar nonsingular matrix with $\det(A) > 0$. Then it can be written as the product of at most four positive definite matrices.

Exceptional Case: $A = \alpha I_n$ with $\alpha \in \mathbb{C} \setminus (0, \infty)$ and $\alpha^n \in (0, \infty)$.

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Corollary

If $A = \alpha P$ for some $\alpha \notin [0, \infty)$ and positive definite matrix P, then A is a product of four positive definite matrix, but not fewer.

Suppose A is a product of k positive definite matrices with $k < 4$. Then $\alpha I_n = P^{-1} A$ is a product of $k + 1$ positive definite matrices.

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Nonsingular matrix

Two positive definite matrices [Ballantine, LAA 3:79-114 3(1970)]

The following are equivalent.

1. A is a product of the product of two positive defnite matrices.
2. A is unitarily similar to an upper block triagnular matrix such that the diagonal blocks are scalar matrices corresponding to distinct positve scalars. with positive diagonal entires.
3. A is similar to a positive diagonal matrices.

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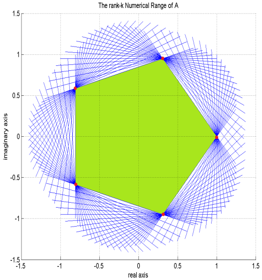
$$(3) \Rightarrow (1): \quad A = S^{-1}DS = (S^{-1}(S^{-1})^*)(S^*DS) = (S^{-1}D(S^{-1})^*)(S^*S)$$

Numerical range

The numerical range of a matrix $A \in M_n$ is defined by

$$W(A) = \{x^* A x : x \in \mathbb{C}^n,\ x^* x = 1\}.$$



- $W(A)$ is always nonempty and convex.
- $W(A) = \{\lambda\}$ if and only if $A = \lambda I$.
- $W(A) \subseteq [0, \infty)$ if and only if $A \geq 0$.
- $W(A) \subseteq \mathbb{R}$ if and only if A is Hermitian.
- $W(A) = \operatorname{conv} \sigma(A)$ if A is normal.
- $W(A) = \bigcap_{t \in [0, 2\pi)} \{\mu \in \mathbb{C} : e^{it}\mu + e^{-it}\bar{\mu} \leq \lambda_1(e^{it}A + e^{-it}A^*)\}$.

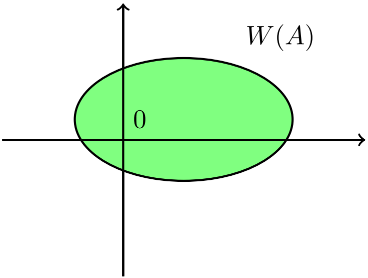
Nonsingular matrix

Three positive definite matrices [Ballantine, LAA 3:79-114 3(1970)]

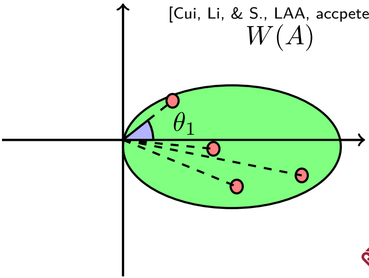
Suppose $A \in M_n$ with $\det(A) > 0$. Then A is the product of three positive definite matrices if and only if one of the following holds.

1. $W(A)$ contains 0 as an interior point.
2. $W(A)$ contains a positive number, and the arguments of the eigenvalues of A can be arranged as

$$-\pi < \theta_1 \le \theta_2 \le \cdots \le \theta_n < \pi \quad \text{such that} \quad \sum_{j=1}^{n} \theta_j = 0.$$



[Cui, Li, & S., LAA, accpeted (201X)]



Singular matrix

P.W. Wu studied the problem for singular case. [Wu, LAA 111:53-61 (1988)]

Singular matrix

Suppose $A \in M_n$ is singular. Then A can be written as the product of at most four positive semidefinite matrices.

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Suppose $A \in M_n$ is singular. Then A can be written as the product of at most four positive semidefinite matrices.

At most four positive semidefinite matrices

Suppose A is non-scalar matrix with $\det(A) \geq 0$. Then it can be written as the product of at most four positive semi-definite matrices.

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The following are equivalent.

- A is a product of the product of two positive semi-defnite matrices.
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Nilpotent

All nonzero nilpotent marices **cannot** be written as the product of two positive semi-definite matrices.

Singular matrix

Given any $A \in M_n$, then A is unitarily similar to

$$\begin{bmatrix} T_1 & R \\ 0 & T_2 \end{bmatrix} \quad \text{such that } T_1 \text{ is invertible and } T_2 \text{ is nilpotent.}$$

Without loss of generality, we may assume that

$$A = \begin{bmatrix} T_1 & R \\ 0 & T_2 \end{bmatrix}.$$

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Block form

Let $B = \begin{bmatrix} B_1 & * \\ 0 & B_2 \end{bmatrix}$. If B is the product of two positive semi-definite matrices, so are B_1 and B_2.

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Singular matrix [Wu, LAA 111:53-61 (1988)]

Suppose $A = \begin{bmatrix} T_1 & R \\ 0 & T_2 \end{bmatrix}$. where T_1 is invertible and T_2 is nilpotent. If $T_2 \neq 0$, then A is a product of at least three positive semi-definite matrices.

Singular matrix

Three positive semi-definite matrices [Wu, LAA 111:53-61 (1988)]

Suppose $A = \begin{bmatrix} T_1 & R \\ 0 & T_2 \end{bmatrix}$. where T_1 is invertible and T_2 is nilpotent.

If T_1 is a product of three positive semi-definite matrices, then A is the product of three positive semi-definite matrices.

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If T_1 is a product of three positive semi-definite matrices, then A is the product of three positive semi-definite matrices.

Remark: Wu conjectured that the converse is also true.

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If T_1 is a product of three positive semi-definite matrices, then A is the product of three positive semi-definite matrices.

Remark: Wu conjectured that the converse is also true.

Counter-example: $A = \begin{bmatrix} -9I_2 & -9I_2 \\ 0_2 & 0_2 \end{bmatrix} \in M_4$. Then

$$A = \begin{bmatrix} 9I_2 & 3I_2 \\ 3I_2 & 2I_2 \end{bmatrix} \begin{bmatrix} 13I_2 & -15I_2 \\ -15I_2 & 18I_2 \end{bmatrix} \begin{bmatrix} I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}.$$

But $T_1 = \begin{bmatrix} -9I_2 \end{bmatrix}$ is the product of five positive definite matrices but not fewer.

Question: So what if T_1 cannot be written as a product of three positive semi-definite matrices?

Singular matrix

Three positive semi-definite matrices [Cui, Li, & S. LAA, accepted (2016)]

Suppose $A = \begin{bmatrix} T_1 & R \\ 0 & T_2 \end{bmatrix}$ where T_1 is invertible and T_2 is nilpotent. Then A is a produt of three positive semi-definite matrices if and only if one of the following holds.

1. $R \neq 0$.
2. $T_2 \neq 0$.
3. $R = 0$ and $T_2 = 0$ and T_1 is the product of three positive definite matrices.

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Some idea: The case when $R \neq 0$ and $T_2 = 0$.

$$T = \begin{bmatrix} I_m & S^* \\ 0 & I_p \end{bmatrix} \begin{bmatrix} T_1 & R \\ 0 & 0_p \end{bmatrix} \begin{bmatrix} I_m & 0 \\ S & I_p \end{bmatrix} = \begin{bmatrix} T_1 + RS & R \\ 0 & 0_p \end{bmatrix}$$

so that $W(T_1 + RS)$ contains 0 as an interior point.

Summary

Let $A \in M_n$ with $\det(A) \geq 0$. Define

$$k = \min\{k : A = P_1 P_2 \cdots P_k \text{ with positive semi-definite } P_j\}.$$

(1) If $A = \alpha I_n$.

  (1.a) If $\alpha \in [0, \infty)$, then $k = 1$.

  (1.b) If $\alpha \notin [0, \infty)$, then $k = 5$.

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(1.a) If $\alpha \in [0, \infty)$, then $k = 1$.

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(2) If (1) does not holds, apply a unitary similarity to A and get

$$\begin{bmatrix} T_1 & R \\ 0 & T_2 \end{bmatrix} \quad \text{such that } T_1 \text{ is invertible and } T_2 \text{ is nilpotent.}$$

(2.a) If $R \neq 0$ or $T_2 \neq 0$, then $k = 3$.

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$(2.\mathrm{a})$ If $R \neq 0$ or $T_2 \neq 0$, then $k = 3$.

$(\mathrm{a.b})$ If $R = 0$, $T_2 = 0$ and T_1 is similar to a positive diagonal matrix, then $k = 2$.

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- (2.a) If $R \neq 0$ or $T_2 \neq 0$, then $k = 3$.
- (a.b) If $R = 0$, $T_2 = 0$ and T_1 is similar to a positive diagonal matrix, then $k = 2$.
- (a.c) If $R = 0$, $T_2 = 0$ and one of the following holds.
  1. $W(T_1)$ contains 0 as an interior point.
  2. $W(T_1)$ contains a positive number, and the arguments of the eigenvalues of T_1 can be arranged as

$$-\pi < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n < \pi \quad \text{such that} \quad \sum_{j=1}^{n} \theta_j = 0.$$

  Then $k = 3$.

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(2) If (1) does not holds, apply a unitary similarity to A and get

$$\begin{bmatrix} T_1 & R \\ 0 & T_2 \end{bmatrix} \quad \text{such that } T_1 \text{ is invertible and } T_2 \text{ is nilpotent.}$$

(2.a) If $R \neq 0$ or $T_2 \neq 0$, then $k = 3$.

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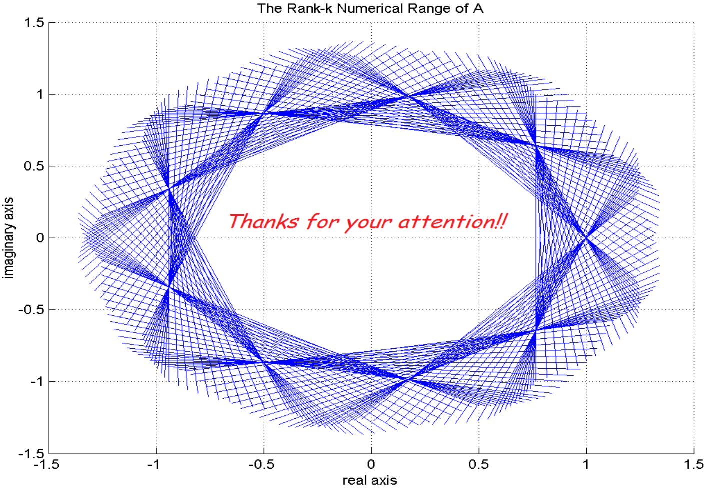
Then $k = 3$.

(3) If none of the above holds, then $k = 4$.



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