Doubly Stochastic Matrices and Majorization

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Majorization

Definition

Let $x = (x_1, x_2, ..., x_n); (y_1, y_2, ..., y_n) \in \mathbb{R}^n$
$(x_1, x_2, ..., x_n)$ is majorized ($\prec$) by $(y_1, y_2, ..., y_n)$
if $\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}$ for all k; $1 \leq k \leq n-1$
(arrows means reordered in descending order) and
$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j$.

Majorization as a partial order

Example

$(2,2,2) \prec (3,2,1) \prec (4,2,0) \prec (6,0,0)$.

Example

$(4,1,1)$ and $(3,3,0)$ are incomparable with respect to the majorization order.

Theorem

Let I be any interval in $\mathbb{R}$ and let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be n-tuples of real numbers in I. Then $a \prec b$ if and only if there exists an n by n doubly stochastic matrix S such that $a = Sb$.

Multivariate and Directional Majorization

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(Multivariate Majorization) Let V be a real vector space and let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be n-tuples of elements of V, then we say that $a \prec b$ if there exists a doubly stochastic matrix S such that $a = Sb$.

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(Directional Majorization) Let V be a real vector space and let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be n-tuples of elements of V, then we say that $a \prec_D b$ if
$(f(a_1), f(a_2), ..., f(a_n)) \prec (f(b_1), f(b_2), ..., f(b_n))$ for all $f \in V^*$.

Remark

For our purposes, we will take $V = \mathbb{C}$ which is a two-dimensional real vector space. Here $\mathbb{C}^*$ up to normalization consists of elements of the form $f(z) = Re(e^{i\theta} z)$.

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Proposition

Majorization in $\mathbb{C}$ is useful in the study of the geometry of polynomials. An example is the following generalization of the Gauss-Lucas theorem: if $p(z) = \prod_{k=1}^{n}(z - z_k)$ and $\bar{z} = \frac{1}{n}\sum_{k=1}^{n} z_k$. If $w_1, w_2, ..., w_{n-1}$ are the zeros of $p'(z)$, then $(w_1, w_2, ..., w_{n-1}, \bar{z}) \prec (z_1, z_2, ..., z_n)$.

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Example

(Martinez, Massey and Silvestre, 2005) The converse is false. Let $a = (1, i, -1, -i)$ and $b = (0, -1 + \frac{3}{2} i, -1 - \frac{3}{2} i, 2)$. Then $a \prec_D b$ but $a \not\prec b$.

Theorem

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Let $\leq$ be a partial order on $\mathbb{C}^n$. Then $\lambda \in \mathbb{C}^n$ is called an eigenvalue of $\leq$ if there exists a nonzero $v \in \mathbb{C}^n$ such that $\lambda v \leq v$. We label the set of all eigenvalues of $\leq$, $spec(\leq)$.

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Observation

It is clear that $spec(\prec) \subseteq spec(\prec_D)$. Do we have equality? (Yes for $n \leq 4$).

Convex Hull Inclusion

Definition

(Convex Hull Inclusion) Let V be a real vector space and let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be n-tuples of elements of V, then we say that $a \prec_C b$ if $conv\{a_i\}_{i=1}^n \subseteq conv\{b_i\}_{i=1}^n$. On $\mathbb{C}^n$, $spec(\prec_C)$ is the set of all possible eigenvalues of n by n stochastic matrices.

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Theorem

(Gauss-Lucas Theorem) If $p(z) = \prod_{k=1}^n (z - z_k)$ and $\bar{z} = \frac{1}{n} \sum_{k=1}^n z_k$. If $w_1, w_2, ..., w_{n-1}$ are the zeros of $p'(z)$, then $(w_1, w_2, ..., w_{n-1}, \bar{z}) \prec_C (z_1, z_2, ..., z_n)$.

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Remark

In 1949, Karpilevich gave a characterization of the the set of all complex numbers which can be the eigenvalue of an n by n stochastic matrix. But what about the set of all complex numbers which can be the eigenvalue of an n by n doubly stochastic matrix?

The Perfect-Mirsky conjecture

Conjecture

(Perfect-Mirsky, 1965) Let Π_k be the k-gon in $\mathbb{C}$ formed by taking the convex hull of all of the kth roots of unity. Then the set of all eigenvalues of n by n doubly stochastic matrices is $\bigcup_{k=1}^{n} \Pi_k$.

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Proposition

The Perfect-Mirsky conjecture is true for $n = 2, 3$ (Perfect-Mirsky, 1965) and for $n = 4$ (Levick-P.-Kribs, 2014) but false for n=5 (Mashreghi-Rivard, 2007) and open for $n \geq 6$.

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Remark

We proved the $n = 4$ case by showing that $spec(\prec_D) = \bigcup_{k=1}^{4} \Pi_k$. So sometimes $spec(\prec_D)$ is easier to work with than $spec(\prec)$.

A new conjecture

Observation

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Conjecture

Let $\lambda \in \mathbb{C}$, then λ is an eigenvalue of an n by n doubly stochastic matrix if and only if either $\lambda \in \Pi_n$ or λ is an eigenvalue of an $n-1$ by $n-1$ stochastic matrix. This conjecture implies that $spec(\prec) = spec(\prec_D)$.

G-Majorization

Definition

Let G be a subgroup of the symmetric group S_n, then we say that $a = (a_1, a_2, ..., a_n)$ is G-majorized by $b = (b_1, b_2, ..., b_n)$ if $(a_1, a_2, ..., a_n)$ is in the convex hull of $\{(b_{\sigma(1)}, b_{\sigma(2)}, ..., b_{\sigma(n)})\}_{\sigma \in G}$.

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Theorem

If G is Abelian, then by the Fundamental theorem of Abelian groups, there exists natural numbers $m_1, m_2, ..., m_k$ with m_t dividing m_{t+1} for all t such that $G = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times ... \times \mathbb{Z}_{m_k}$. In this case $spec(\prec_G)$ is the m_k-gon whose vertices are the m_k-th roots of unity.

Question

When is $spec(\prec_G)$ a union of regular polygons with vertices on the unit circle? (Yes for G Abelian or $G = S_3$ or $G = S_4$, no for $G = S_5$.) Does the solvability of G play a role in this question?

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Question

When is $spec(\prec_G) = \bigcup_H spec(\prec_H)$ where the union is taken over the set of all subgroups H of G having property Q for some choice of Q?

Remark

If property Q is cyclic this question is equivalent to the first one. Other choices of property Q include proper and normal or just proper.

Connection with group representations

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Observation

Let G be a finite group and $g \to U_g$ be an irreducible unitary group representation of G. Let $C = conv\{U_g : g \in G\}$. Call the set of all eigenvalues of elements of C, the spectral set of this representation. It turns out that $spec(\prec_G)$ will always be a union of spectral sets of certain representations of G. What can we say about the spectral sets? When are they unions of regular polygons with vertices on the unit circle?

Thank You