Separability of three qubit Greenberger-Horne-Zeilinger diagonal states

2016.07.04., Jeju, Korea

Seung-Hyeok Kye (Seoul National University)
a joint work with Kyung Hoon Han

1. Greenberger-Horne-Zeilinger diagonal states

A state on the Hilbert space $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$ is said to be *(fully) separable* if it is the convex combination

$$\varrho := \sum_{i \in I} p_i |z_i\rangle\langle z_i|$$

of pure product states $|z_i\rangle\langle z_i|$ onto product vectors $|z_i\rangle$:

$$|z_i\rangle = |x_{1i}\rangle \otimes |x_{2i}\rangle \otimes \cdots \otimes |x_{ni}\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}, \qquad i \in I.$$

Therefore, ϱ is a $d \times d$ density matrix, with the dimension $d = \prod_{i=1}^{n} d_i$ of the Hilbert space $\mathcal{H}$. The set consisting of all separable states is a convex set whose extreme points are pure product states.

A state which is not separable is call *entangled*.

To get entanglement, we begin with non-product vector, say

$$\begin{aligned}|\xi\rangle &= |00\rangle + |11\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \\ &= |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \\ &= (1,0)^{\rm t} \otimes (1,0)^{\rm t} + (0,1)^{\rm t} \otimes (0,1)^{\rm t} \in \mathbb{C}^2 \otimes \mathbb{C}^2 \\ &= (1,0,0,1)^{\rm t} \in \mathbb{C}^4\end{aligned}$$

and take pure state

$$|\xi\rangle\langle\xi| = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} (1,0,0,1) = \begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix}.$$

The partial transpose (= block-wise transpose):

$$\begin{pmatrix} 1\cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}.$$

is not positive (=positive semi-definite).

- In general, $\varrho \in M_m \otimes M_n$ is separable $\Longrightarrow$ PPT.
  Choi (1982), Peres (1996)
- The converse holds if and only if $(m, n) = (2, 2), (2, 3)$ or $(3, 2)$.
  Størmer (1963), Woronowicz (1976), Choi (1982), Horodecki's (1996)

It is extremely difficult in general to distinguish entanglement from separability.
Actually, it is known to be an NP-hard problem [Gurvits (2004)].

The three qubit GHZ state basis consists of eight vectors in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ given by

$$|\xi_{ijk}\rangle = \frac{1}{\sqrt{2}}\left(|i\rangle \otimes |j\rangle \otimes |k\rangle + (-1)^i |\bar{i}\rangle \otimes |\bar{j}\rangle \otimes |\bar{k}\rangle\right), \qquad i + \bar{i} = 1 \mod 2$$

where the index ijk runs for $i, j, k \in \{0, 1\}$. We may write down as follows:

$$\begin{aligned}
|000\rangle + |111\rangle &= (1,0,0,0,0,0,0,1)\\
|001\rangle + |110\rangle &= (0,1,0,0,0,0,1,0)\\
|010\rangle + |101\rangle &= (0,0,1,0,0,1,0,0)\\
|011\rangle + |100\rangle &= (0,0,0,1,1,0,0,0)\\
|100\rangle - |011\rangle &= (0,0,0,-1,1,0,0,0)\\
|101\rangle - |010\rangle &= (0,0,-1,0,0,1,0,0)\\
|110\rangle - |001\rangle &= (0,-1,0,0,0,0,1,0)\\
|111\rangle - |000\rangle &= (-1,0,0,0,0,0,0,1)
\end{aligned}$$

We endow the indices with the lexicographic order to get eight vectors $\xi_1, \xi_2, \dots \xi_8$. A GHZ diagonal state is of the form

$$\varrho = \sum_{i=1}^{8} p_i |\xi_i\rangle\langle\xi_i|$$

for nonnegative p_i's with $\sum_{i=1}^{8} p_i = 1$.

So, ϱ is X-shaped;

$$X(a,b,c) := \begin{pmatrix} a_1 & & & & & & & c_1 \\ & a_2 & & & & & c_2 & \\ & & a_3 & & & c_3 & & \\ & & & a_4 & c_4 & & & \\ & & & \bar{c}_4 & b_4 & & & \\ & & \bar{c}_3 & & & b_3 & & \\ & \bar{c}_2 & & & & & b_2 & \\ \bar{c}_1 & & & & & & & b_1 \end{pmatrix}.$$

$\varrho = \frac{1}{2}X(a, b, c)$ with

$$\begin{aligned} a = b &= (p_1 + p_8,\ p_2 + p_7,\ p_3 + p_6,\ p_4 + p_5), \\ c &= (p_1 - p_8,\ p_2 - p_7,\ p_3 - p_6,\ p_4 - p_5). \end{aligned}$$

Conversely, every X-shaped state $X(a, b, c)$ can be realized as a GHZ diagonal state whenever $a = b$ and $c \in \mathbb{R}^4$.

Goal: Characterize separability of $X(a, a, c)$ when $a_i \geq 0$ and c_i is real.

In three qubit case, there are three kinds of partial transpose. Separability implies positivity of these partial transposes.

Describe GHZ diagonal states, using Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

as

$$\begin{aligned} X(a,a,c) =& \frac{1}{8}(I \otimes I \otimes I + \lambda_2 Z \otimes Z \otimes I + \lambda_3 Z \otimes I \otimes Z + \lambda_4 I \otimes Z \otimes Z \\ & + \lambda_5 X \otimes X \otimes X + \lambda_6 Y \otimes Y \otimes X \\ & + \lambda_7 Y \otimes X \otimes Y + \lambda_8 X \otimes Y \otimes Y), \end{aligned}$$

with the coefficients

$$\begin{aligned} &\lambda_2 = 2(+a_1 + a_2 - a_3 - a_4), \\ &\lambda_3 = 2(+a_1 - a_2 + a_3 - a_4), \qquad && \lambda_4 = 2(+a_1 - a_2 - a_3 + a_4), \\ &\lambda_5 = 2(+c_1 + c_2 + c_3 + c_4), && \lambda_6 = 2(-c_1 - c_2 + c_3 + c_4), \\ &\lambda_7 = 2(-c_1 + c_2 - c_3 + c_4), && \lambda_8 = 2(-c_1 + c_2 + c_3 - c_4). \end{aligned}$$

- A. Kay (2011): A GHZ diagonal state is separable if and only if it is of PPT, whenever $\Pi_{i=5}^{8}\lambda_i < 0$.
- O. Gühne (2011): the inequality

$$8\min\{a_1, a_2, a_3, a_4\} \geq \sqrt{\frac{(\lambda_5\lambda_6 + \lambda_7\lambda_8)(\lambda_5\lambda_7 + \lambda_6\lambda_8)(\lambda_5\lambda_8 + \lambda_6\lambda_7)}{\lambda_5\lambda_6\lambda_7\lambda_8}}$$

  is a sufficient criterion for the separability of GHZ diagonal states, when $\Pi_{i=5}^{8}\lambda_i > 0$.

Gühne also gave a necessary condition: Define

$$\begin{aligned}\mathcal{L}(\varrho, z) :=& \mathrm{Re}\left(z_1 c_1 + z_2 c_2 + z_3 c_3 + z_4 \bar{c}_4\right),\\ \mathcal{F}(z) :=& \mathrm{Re}(z_1)\cos(\alpha+\beta+\gamma) - \mathrm{Im}(z_1)\sin(\alpha+\beta+\gamma) + \mathrm{Re}(z_2)\cos(\alpha) - \\ &+ \mathrm{Re}(z_3)\cos(\beta) - \mathrm{Im}(z_3)\sin(\beta) + \mathrm{Re}(z_4)\cos(\gamma) - \mathrm{Im}(z_4)\sin(\gamma),\\ C(z) :=& \sup_{\alpha,\beta,\gamma} |\mathcal{F}(z)|\end{aligned}$$

and showed that that if $\varrho = X(a, b, c)$ is separable then the inequality

$$\mathcal{L}(\varrho, z) \leq C(z)\ \Delta_\varrho$$

holds for every $z \in \mathbb{C}^4$, where the number Δ_ϱ is given by

$$\Delta_\varrho = \min\{\sqrt{a_i b_i}\ (i = 1, 2, 3, 4),\ \sqrt[4]{a_1 b_2 b_3 a_4},\ \sqrt[4]{b_1 a_2 a_3 b_4}\}$$

which is determined by the diagonal entries of $\varrho = X(a, b, c)$.

Note: $\Delta_\varrho = \min\{a_i\ (i = 1, 2, 3, 4)\}$, when ϱ is a GHZ diagonal state.

We show:

- Gühne's necessary condition is also sufficient.
  We get the formula of $C(z)$ in terms of variables z_i's.
- In case $\Pi_{i=5}^{8}\lambda_i \geq 0$, we find the dichotomy
  with a condition $[\star]$ in terms of anti-diagonal entries c_i's:
  $\triangleright$ If $[\star]$ does not hold, then separable = PPT.
  $\triangleright$ If $[\star]$ holds, then Gühne's sufficient condition is also necessary.

Our main tool is to use the duality between tri-tensor product and bi-linear maps.

2. Entanglement witnesses

For a multi-linear map ϕ from $M_{d_1} \times \cdots \times M_{d_{n-1}}$ into M_{d_n}, we associate a matrix $C_\phi \in M_d$ by

$$C_\phi = \sum_{i_1, j_1, \ldots, i_{n-1}, j_{n-1}} |i_1\rangle\langle j_1| \otimes \cdots \otimes |i_{n-1}\rangle\langle j_{n-1}| \otimes \phi(|i_1\rangle\langle j_1|, \cdots, |i_{n-1}\rangle\langle j_{n-1}|)$$
$$\in M_{d_1} \otimes \cdots \otimes M_{d_{n-1}} \otimes M_{d_n}$$

The correspondence $\phi \mapsto C_\phi$ is nothing but the Choi-Jamiołkowski isomorphism for bi-partite case of $n = 2$.

We also define the bilinear pairing between ϱ in $M_{d_1} \otimes \cdots \otimes M_{d_{n-1}} \otimes M_{d_n}$ and a multi-linear map $\phi : M_{d_1} \times \cdots \times M_{d_{n-1}} \to M_{d_n}$ by

$$\langle \varrho, \phi \rangle = \langle \varrho, C_\phi \rangle := \mathrm{Tr}\,(C_\phi\, \varrho^{\mathrm{t}}).$$

Theorem (K, J. Phys. A 48 (2015), 235303)

An n-partite state ϱ is (fully) separable if and only if $\langle \varrho, \phi \rangle \geq 0$ for every positive $(n-1)$-linear map ϕ.

Here, ϕ is positive if

$$x_1 \in M_{d_1}^+, \ldots, x_{n-1} \in M_{d_{n-1}}^+ \implies \phi(x_1, \ldots, x_{n-1}) \in M_{d_n}^+$$

- M. Horodecki, P. Horodecki and R. Horodecki (1996): $n = 2$,
- M. Eom + K (2000): $n = 2$, to find the dual cone of positive maps.

B. M. Terhal (2000)
A non-positive self-adjoint $d \times d$ matrix W is entanglement witness if

- $\langle \varrho, W \rangle \geq 0$ for every separable state ϱ.

Then, we have

- every entangled state ϱ is 'detected' by an entanglement witness W in the sense: $\langle \varrho, W \rangle < 0$.
- every entanglement witness W detects an entangled state.

We have interpreted entanglement witnesses as separate or coordinate-wise positivity of multi-linear maps, which is the weakest notion of positivity.

Observation: Let ϱ be an X-shaped state. Then, ϱ is separable if and only if $\langle \varrho, W \rangle \geq 0$ for every X-shaped entanglement witness W.

For a given $(s,t) \in \mathbb{R}_+^4 \times \mathbb{R}_+^4$ and $u \in \mathbb{C}^4$, we introduce two numbers:

$$A(s,t) = \inf_{r>0} \left[\sqrt{(s_1 r^{-1} + t_4 r)(s_4 r^{-1} + t_1 r)} + \sqrt{(s_2 r^{-1} + t_3 r)(s_3 r^{-1} + t_2 r)} \right]$$

$$B(u) = \max_{\theta} \left(|u_1 e^{\mathrm{i}\theta} + \bar{u}_4| + |u_2 e^{\mathrm{i}\theta} + \bar{u}_3| \right).$$

Here, $\mathbb{R}_+$ denotes $[0, \infty)$. Then we see that a three qubit non-positive self-adjoint matrix $W = X(s,t,u)$ is an entanglement witness if and only if the inequality $A(s,t) \geq B(u)$ holds [Han + K, J. Math. Phys. (2016)].

Theorem

For $z \in \mathbb{C}^4$, we have $C(z) = B(z_1, z_2, z_3, \bar{z}_4)$.

Recall that $C(z)$ was defined as the maximum of a three variable function.

We say that an X-shaped self-adjoint matrix $W = X(s,t,u)$ is GHZ diagonal if $s = t$ and $u \in \mathbb{R}$.

Theorem

Let ϱ be a GHZ diagonal state. Then, ϱ is separable if and only if $\langle \varrho, W \rangle \geq 0$ for every GHZ diagonal witness W.

Theorem

Let $\varrho = X(a,a,c)$ be a GHZ diagonal state. Then, ϱ is separable if and only if the inequality

$$\mathcal{L}(\varrho, z) \leq C(z)\ \Delta_\varrho$$

holds for every $z \in \mathbb{R}^4$.

3. Entry-wise characterization

Look for maximum of

$$f(z_1, z_2, z_3, z_4) := \frac{\mathcal{L}(\varrho, z)}{C(z)} \quad = \frac{c_1 z_1 + c_2 z_2 + c_3 z + c_4 z_4}{\max_\theta \left(|z_1 e^{\mathrm{i}\theta} + z_4| + |z_2 e^{\mathrm{i}\theta} + z_3|\right)}$$

with real $z_i's$ and c_i's.

The function f enjoys:

- $f(\alpha z) = f(z)$ for $\alpha > 0$
- $f(-z) = -f(z)$.

We partition the domain of f by $\mathbb{R}^4 \setminus \{0\} = \Omega^+ \sqcup \Omega^-$ with

$$\Omega^+ = \{z \in \mathbb{R}^4 \setminus \{0\} : z_1 z_2 z_3 z_4 \geq 0\}, \;\; \Omega^- = \{z \in \mathbb{R}^4 \setminus \{0\} : z_1 z_2 z_3 z_4 < 0\}.$$

We also partition Ω^- by Ω_i^- $(i = 0, 1, 2, 3, 4)$

$$\Omega_i^- = \left\{ z \in \Omega^- : \tfrac{1}{|z_i|} \geq \textstyle\sum_{j \neq i} \tfrac{1}{|z_j|} \right\}, \qquad i = 1, 2, 3, 4,$$
$$\Omega_0^- = \Omega^- \setminus \left(\sqcup_{i=1}^4 \Omega_i^- \right) = \left\{ z \in \Omega^- : \tfrac{1}{|z_i|} < \textstyle\sum_{j \neq i} \tfrac{1}{|z_j|}, \; i = 1, 2, 3, 4 \right\}.$$

$z \in \Omega_0^-$ if and only if the reciprocals of entries make a quadrangle.

Theorem

For $z \in \mathbb{R}^4 \setminus \{0\}$, we have

$$C(z) = \begin{cases} |z_1| + |z_2| + |z_3| + |z_4|, & z \in \Omega^+, \\ |z_1| + |z_2| + |z_3| + |z_4| - 2|z_i|, & z \in \Omega_i^- \;(i = 1, 2, 3, 4), \\ \sqrt{\frac{(z_1 z_2 - z_3 z_4)(z_2 z_4 - z_1 z_3)(z_1 z_4 - z_2 z_3)}{-z_1 z_2 z_3 z_4}}, & z \in \Omega_0^-. \end{cases}$$

Consider only one component of Ω^-, say $z_1 < 0,\ z_i > 0\ (i = 2, 3, 4)$, and restrict the domain of f by $\sum_{i=1}^{4} |z_i| = 1$

- The region is the tetrahedron with four extreme points $-E_1, E_2, E_3$ and E_4.
- Ω_0^- occupies the central part of the tetrahedron
  touches four extreme points and six edges
  does not touch four faces
- the boundary between Ω_0^- and Ω_i^-, say Ω_1^-
  is a two-dimensional surface in the tetrahedron
  touches the boundary of the face $(=\triangle E_2 E_3 E_4)$ opposite to $-E_1$
  is far from $-E_1$ and near from the $\triangle E_2 E_3 E_4$

Global behavior of the function f:

- The maximum of f on Ω^+ is $\max |c_i|$.
- No extreme value occurs on the interior of Ω_i^- for $i = 1, 2, 3, 4$.
- f is not differentiable on the boundary between Ω_0^- and Ω_i^- for $i = 1, 2, 3, 4$.
- On this boundary(=two dimensional surface), no extreme value occurs possibly except on a curve where the value of f is $\pm c_i$.

It remains to consider the case the maximum of f occurs on the interior of Ω_0^-.

If an extreme value of f occurs on $s = (s_1, s_2, s_3, s_4) \in \Omega_0^-$, then $t_i = -\frac{1}{s_i}$ is given by

$$
\begin{aligned}
t_1 &= c_1(+c_1^2 - c_2^2 - c_3^2 - c_4^2) + 2c_2c_3c_4, \\
t_2 &= c_2(-c_1^2 + c_2^2 - c_3^2 - c_4^2) + 2c_1c_3c_4, \\
t_3 &= c_3(-c_1^2 - c_2^2 + c_3^2 - c_4^2) + 2c_1c_2c_4, \\
t_4 &= c_4(-c_1^2 - c_2^2 - c_3^2 + c_4^2) + 2c_1c_2c_3,
\end{aligned}
$$

with the critical value

$$
\begin{aligned}
f(s_1, s_2, s_3, s_4)^2 &= \frac{(\lambda_5\lambda_6 + \lambda_7\lambda_8)(\lambda_5\lambda_7 + \lambda_6\lambda_8)(\lambda_5\lambda_8 + \lambda_6\lambda_7)}{8^2\lambda_5\lambda_6\lambda_7\lambda_8} \\
&= \frac{(c_1c_2 - c_3c_4)(c_1c_3 - c_2c_4)(c_1c_4 - c_2c_3)}{\lambda_5\lambda_6\lambda_7\lambda_8},
\end{aligned}
$$

which appears in the sufficient condition of Gühne.

Using the relation

$$(t_2 + t_3)^2 - (t_1 + t_4)^2 = \lambda_5\lambda_8(\lambda_6\lambda_7)^2,$$
$$(t_1 - t_4)^2 - (t_2 - t_3)^2 = \lambda_6\lambda_7(\lambda_5\lambda_8)^2.$$

One may conclude that the function f has a critical value on Ω_0^- if and only if the the inequality $\lambda_5\lambda_6\lambda_7\lambda_8 > 0$ together with the condition [$\star$]:

$$t_1 t_4\lambda_6\lambda_7 < 0 \quad \text{and} \quad t_2 t_3\lambda_5\lambda_8 > 0 \qquad [\star]$$

holds.

Theorem

Let $\varrho = X(a, a, c)$ be a GHZ diagonal state with $\lambda_5, \lambda_6, \lambda_7, \lambda_8$ and t_1, t_2, t_3, t_4 given by c_i. Then we have the following:

(i) *if $\lambda_5\lambda_6\lambda_7\lambda_8 \le 0$, then ϱ is separable if and only if it is of PPT,*

(ii) *if $\lambda_5\lambda_6\lambda_7\lambda_8 > 0$ and [$\star$] does not hold, then ϱ is separable if and only if it is of PPT,*

(iii) *if $\lambda_5\lambda_6\lambda_7\lambda_8 > 0$ and [$\star$] holds, then ϱ is separable if and only if the inequality*

$$8\min\{a_1, a_2, a_3, a_4\} \ge \frac{\sqrt{(\lambda_5\lambda_6 + \lambda_7\lambda_8)(\lambda_5\lambda_7 + \lambda_6\lambda_8)(\lambda_5\lambda_8 + \lambda_6\lambda_7)}}{\sqrt{\lambda_5\lambda_6\lambda_7\lambda_8}}$$

holds.

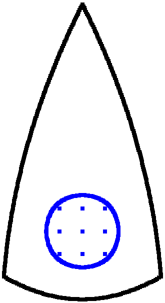
4. Other kinds of separability

In multi-partite case, there are several kinds of separability.

In the tri-partite case, a state $\varrho \in M_A \otimes M_B \otimes M_C$ may be considered as a bi-partite states by

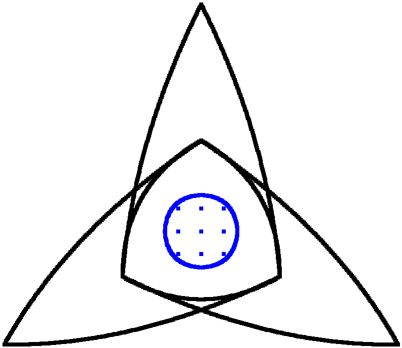
$$\varrho \in M_A \otimes M_B \otimes M_C = M_A \otimes (M_B \otimes M_C) = M_A \otimes M_{BC}.$$

Then, we have the notion of A-BC separability as well as full separability.



fully separable

A-BC separable

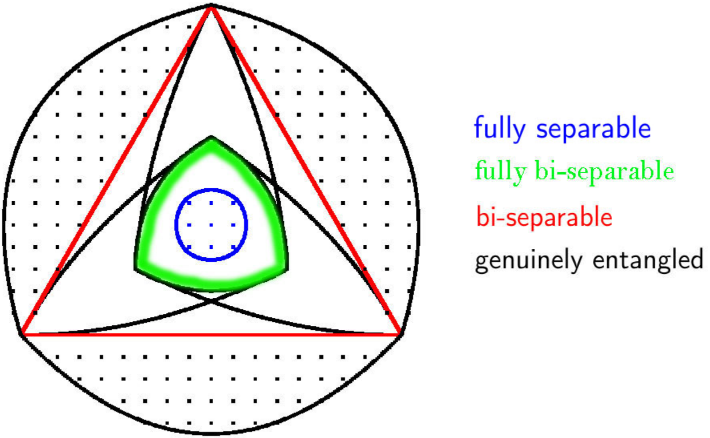


fully separable

A-BC separable

B-CA separable

C-AB separable


fully separable
fully bi-separable
bi-separable
genuinely entangled

In general, we have the relations:

$$
\begin{array}{ccccc}
\text{fully separable} & \Longrightarrow & \text{fully bi-separable} & \Longrightarrow & \text{bi-separable} \\
 & & \color{blue}{\Downarrow} & & \color{blue}{\Downarrow} \\
 & & \text{PPT} & \Longrightarrow & \text{PPT mixture} \\
 & & & & \Downarrow \\
 & & & & \text{state}
\end{array}
$$

Blue arrows become equivalence relations for X-states.

Consider the following X-shaped n-qubit cases:

$$
\varrho = \begin{pmatrix}
a_{00\ldots0} & & & & & & & & & c_{00\ldots0} \\
 & \ddots & & & & & & & \iddots & \\
 & & a_{\mathbf{i}} & & & & & c_{\mathbf{i}} & & \\
 & & & \ddots & & & \iddots & & & \\
 & & & & a_{01\ldots1} & c_{01\ldots1} & & & & \\
 & & & & \bar{c}_{01\ldots1} & b_{01\ldots1} & & & & \\
 & & & \iddots & & & \ddots & & & \\
 & & \bar{c}_{\mathbf{i}} & & & & & b_{\mathbf{i}} & & \\
 & \iddots & & & & & & & \ddots & \\
\bar{c}_{00\ldots0} & & & & & & & & & b_{00\ldots0}
\end{pmatrix},
$$

with $\mathbf{i}$ is an n-multi-index, expressed by an n string of 0,1 beginning with 0.

Theorem (K.-H. Han + K, J. Phys. A 49 (2016), 175303)

For a multi-qubit $\mathbf{X}$*-state* ϱ*, we have the following:*

- ϱ *is fully bi-separable if and only if it is of PPT if and only if* $a_{\mathbf{i}} b_{\mathbf{i}} \geq |c_{\mathbf{j}}|^2$ *for every* $\mathbf{i}$ *and* $\mathbf{j}$
- ϱ *is bi-separable if and only if it is a PPT-mixture if and only if* $\sum_{\mathbf{j} \neq \mathbf{i}} \sqrt{a_{\mathbf{j}} b_{\mathbf{j}}} \geq |z_{\mathbf{i}}|$ *for each* $\mathbf{i}$*.*

The equivalence between bi-separability and inequality was shown by O. Gühne and M. Seevinck (2010), T. Gao and Y. Hong (2011).

The arguments of anti-diagonal entries play no role in this cases.

감사합니다.