Hyperbolic geometry of positive definite matrices associated with geometric mean

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Geometric mean

The geometric mean of $a, b > 0$ is

$$\sqrt{ab}.$$

$\mathbb{P}_n$ = the set of $n \times n$ positive definite matrices over $\mathbb{C}$ or $\mathbb{R}$.

Generally speaking, $\mathbb{P}_n$ "behaves" like $\mathbb{R}_+$ (the set of positive numbers).

Question: What is a "good" definition of the geometric mean, denoted by $A\#_{\frac{1}{2}}B$, of $A, B \in \mathbb{P}_n$?

The geometric mean of A and B (Pusz and Woronowicz 1975):

$$A\#_{\frac{1}{2}}B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}.$$

When $n = 1$, $A = a, B = b$, we have $a\#_{\frac{1}{2}}b = \sqrt{ab}$. One can check

$$A\#_{\frac{1}{2}}B = B\#_{\frac{1}{2}}A.$$

• W. Pusz and S. L. Woronowicz, Functional calculus for sesquilinear forms and the purification map, Rep. Math. Phys., 8 (1975) 159-170.

• T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, Linear Algebra Appl., 26 (1979) 203-241.

$\mathbb{P}_n$ is a Riemannian manifold

$\mathbb{P}_n$ is a Riemannian manifold equipped with the Riemannian structure:

$$Q_p(X, Y) = \mathrm{tr}\,(p^{-1} X p^{-1} Y)$$

where $X, Y \in T_p(\mathbb{P}_n)$. Let $\gamma : [\alpha, \beta] \to \mathbb{P}_n$ be a curve. The arc length of γ is

$$L(\gamma) := \int_\alpha^\beta \|\dot\gamma(t)\| dt,$$

where $\|\dot\gamma(t)\| = Q^{1/2}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))$. The Riemannian structure turns $\mathbb{P}_n$ into a metric space:

$$d(A, B) := \inf_\gamma L(\gamma) = \left(\sum_{i=1}^n \log^2 \lambda_i(BA^{-1}) \right)^{1/2}$$

$A\#_{\frac{1}{2}}B$ is the mid-point of the geodesic joining A and B

Theorem (Bhatia and Holbrook 2006)

The unique geodesic joining A and B in $\mathbb{P}_n$ has the parametrization

$$\gamma(t) = A\#_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}, \qquad 0 \le t \le 1.$$

So $A\#_{\frac{1}{2}}B$ is the midpoint of the geodesic joining A and B.

- R. Bhatia and J.A.R. Holbrook, Riemannian Geometry and Matrix Geometric Means, Linear Alg. App. 413 (2006) 594-618.

- R. Bhatia, *Positive Definite Matrices*, Princeton University Press, 2007.

Log majorization

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$. Let $x^{\downarrow} = (x_{[1]}, x_{[2]}, \ldots, x_{[n]})$ denote a rearrangement of the components of x such that $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. We say that x is *majorized* by y, denoted by $x \prec y$, if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

Weak majorization: $x \prec_w y$ means $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$, $k = 1, 2, \ldots, n$.
If $x > 0$ (i.e., $x_i > 0$ for $i = 1, \ldots, n$) and $y > 0$, we say that x is *log majorized* by y, denoted by $x \prec_{\log} y$, if

$$\prod_{i=1}^{k} x_{[i]} \leq \prod_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1, \quad \text{and} \quad \prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}.$$

In other words, $x \prec_{\log} y$ if and only if $\log x \prec \log y$.

Inequalities of Ando-Hiai and Araki

Let $A, B \in \mathbb{P}_n$. For any $t \in [0, 1]$ and $s > 0$,

$$
\begin{aligned}
\lambda(A \#_t B) &\prec_{\log} \lambda\left(e^{(1-t)\log A + t\log B}\right) && \text{(Ando and Hiai, 1994)} \\
&\prec_{\log} \lambda\left(B^{ts/2} A^{(1-t)s} B^{ts/2}\right)^{1/s} && \text{(Araki, 1990)} \\
&= \lambda\left(A^{(1-t)s} B^{ts}\right)^{1/s}.
\end{aligned}
$$

Inequalities of Ando-Hiai and Hiai-Petz

Theorem (Ando and Hiai 1994)

For any $A, B \in \mathbb{P}_n$ and $0 \le r \le 1$,

$$(A \#_t B)^r \prec_{\log} A^r \#_t B^r.$$

or equivalently

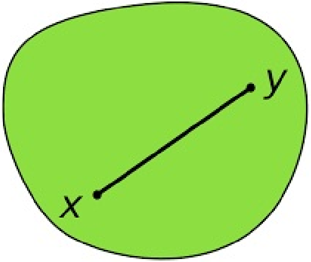
$$(A^p \#_t B^p)^{1/p} \prec_{\log} (A^q \#_t B^q)^{1/q}, \quad 0 < q \le p.$$

Theorem (Hiai and Petz 1993)

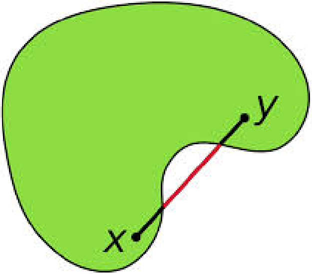
For any $0 \le t \le 1$ and H, K Hermitian,

$$\exp\{(1-t)H + tK\} = \lim_{r \to 0} \{\exp(rH) \#_t \exp(rK)\}^{1/r}$$

Convex body in Euclidean space



convex



nonconvex

For any given $A \in \mathbb{P}_n$, define

$$M(A) := \{B \in \mathbb{P}_n : \lambda(B) \prec_{\log} \lambda(A)\} \subset \mathbb{P}_n.$$

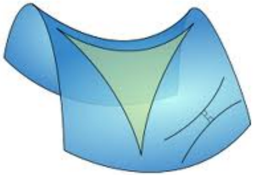
The exponential map $\exp : \mathbb{H}_n \to \mathbb{P}_n$ is a diffeomorphism so $\log : \mathbb{P}_n \to \mathbb{H}_n$ is defined. The set $M(A)$ is convex in the multiplicative sense since

$$\log(M(A)) = \{H : H \in \mathbb{C}_{n\times n} \text{ is Hermitian and } \lambda(H) \prec \lambda(\log A)\}$$

is a convex set. Indeed, it is equal to the convex hull of the set consisting of all Hermitian matrices with spectrum majorized by $\lambda(\log(A))$. However, one can easily see that $M(A)$ is not closed under matrix addition. So $M(A)$ is not convex in $\mathbb{P}_n$ when $\mathbb{P}_n$ is viewed as a subset of the Euclidean space $\mathbb{C}_{n\times n}$, in which $\mathbb{P}_n$ is a cone.

Geodesically convex

We say that $C \subset \mathbb{P}_n$ is geodesically convex if all geodesics between any two points lie in $\mathbb{P}_n$. Similarly we define the geodesic convex hull of a subset S in $\mathbb{P}_n$ to be the smallest geodesically convex set that contains S.



$M(A)$ is geodesically convex

Theorem (DAH, 2016)

Let $A \in \mathbb{P}_n$. The set

$$M(A) = \{B \in \mathbb{P}_n : \lambda(B) \prec_{\log} \lambda(A)\} \subset \mathbb{P}_n$$

is geodesically convex with respect to the Riemannian structure of $\mathbb{P}_n$. In other words, if $B, C \in M(A)$, then the geodesic joining B and C lies in $M(A)$. So

$$M(A) = \{A \natural_t B : t \in [0, 1], B \in \mathbb{P}_n, \lambda(B) \prec_{\log} \lambda(A)\}.$$

- T. H. Dinh, S. Ahsani and T.Y. Tam, Geometry and inequalities of geometric mean, to appear in Czechoslovak Mathematical Journal.

$M(A)$ is the convex hull of an orbit

Theorem (DAT, 2015)

The set $M(A)$ is the geodesic convex hull of the orbit

$$\begin{aligned} O(A) &:= \{UAU^* : U \in \mathrm{U}(n)\} \\ &= \{B \in \mathbb{P}_n : \lambda(B) = \lambda(A)\} \end{aligned}$$

consisting of all $B \in \mathbb{P}_n$ with spectrum coincides with that of A.

Unitarily invariant norm

A norm $|||\cdot|||$ on $\mathbb{C}_{n\times n}$ is said to be *unitarily invariant* if

$$|||UAV||| = |||A||| \quad \text{for all unitary } U, V \in \mathbb{C}_{n\times n}.$$

For example, the spectral norm

$$\|A\| = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

is unitarily invariant.

Denote by $s(A) = (s_1(A), \ldots, s_n(A))$ the vector of singular values of A. A map $\Phi : \mathbb{R}^n \to \mathbb{R}_+$ is called a symmetric gauge function if

1. Φ is a norm.
2. $\Phi(Px) = \Phi(x)$ for all $x \in \mathbb{R}^n$ and $P \in S_n$.
3. $\Phi(\epsilon_1 x_1, \ldots, \epsilon_n x_n) = \Phi(x_1, \ldots, x_n)$, whenever $\epsilon_i = \pm 1$.

So symmetric gauge function is completely determined by its values on $\mathbb{R}^n_+$.

von Neumann's characterization

Theorem (von Neumann)

Given a symmetric gauge function Φ on $\mathbb{R}^n$, the function

$$|||A|||_{\Phi} = \Phi(s(A))$$

defines a unitarily invariant norm on $\mathbb{C}_{n\times n}$. Conversely, if $|||\cdot|||$ is a unitarily invariant norm on $\mathbb{C}_{n\times n}$, then

$$\Phi_{|||\cdot|||}(x) = |||\operatorname{diag} x|||$$

is a symmetric gauge function on $\mathbb{R}^n$.

$$\{\text{unitarily invariant norms}\} \longleftrightarrow \{\text{ symmetric gauge functions}\}$$

• J. von Neumann, Some matrix inequalities and metrication of matrix space, Tomsk. Univ. Rev., **1** (1937) 286-300.

Ky Fan Dominance Theorem

Theorem (Ky Fan's Dominance Theorem)

Let $A, B \in \mathbb{C}_{n\times n}$. Then $|||A||| \leq |||B|||$ for all unitarily invariant norms $|||\cdot|||$ if and only if $s(A) \prec_w s(B)$, where $s(A)$ denotes the vector of singular values $s_1(A) \geq \cdots \geq s_n(A)$ of A.

Corollary (DAT, 2016)

If $A, B \in \mathbb{P}_n$ such that $B \prec_{\log} A$, then for all $t \in [0,1]$, we have $A\sharp_t B \prec_{\log} A$, or equivalently, $|||A\sharp_t B||| \leq |||A|||$ for all unitarily invariant norms $|||\cdot|||$ on $\mathbb{C}_{n\times n}$.

Remark: The singular values of A are the square roots of the eigenvalues of A^*A or AA^*.

Theorem (DAT, 2016)

Let $A_i, B_i \in \mathbb{P}_n$, $i = 1, \ldots, m$. For any unitarily invariant norm $|||\cdot|||$ on $\mathbb{C}_{n\times n}$, we have

$$\begin{aligned} & ||| \sum_{i=1}^{m} (A_i \sharp_t B_i)^2 ||| \\ \leq \; & ||| (\sum_{i=1}^{m} A_i)^{1/2} (\sum_{i=1}^{m} B_i) (\sum_{i=1}^{m} A_i)^{1/2} ||| \\ \leq \; & ||| (\sum_{i=1}^{m} A_i)(\sum_{i=1}^{m} B_i) ||| \, . \end{aligned}$$

Bhatia's triangle inequality

Theorem (Bhatia 2003)

Let $A, B, C \in \mathbb{P}_n$. Then

$$d(A\#_t B, A\#_t C) \leq t d(B, C), \quad 0 \leq t \leq 1$$

i.e., d is convex.

- R. Bhatia, On the exponential metric increasing property, Linear Alg. Appl. 375 (2003) 211-220.

Euclidean vs hyperbolic

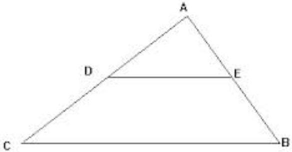


Figure 1 (Euclidean triangle)

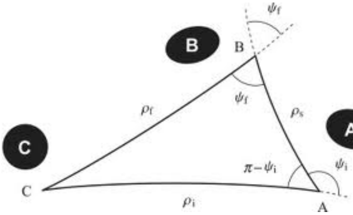


Figure 2 (Hyperbolic geodesic triangle)

where $D = A\#_t B$, $E = A\#_t C$.

It asserts that when the geodesics arcs next to A of the geodesic triangle $\Delta(A, B, C)$ retract at the same pace, the length of the opposite geodesic to A will retract faster, in contrast to the triangle in the ordinary Euclidean space.

However, the geodesic joining $A, B \in P$ contains more information than merely its length $d(A, B)$.

Riemannian symmetric space

A symmetric space is a smooth manifold whose group of symmetries contains an inversion symmetry about every point. One can formulate the inversion symmetry either via Riemannian geometry or via Lie theory. The Lie-theoretic definition is more general and more algebraic.

Riemannian symmetric spaces arise in both mathematics and physics. They were first studied extensively and classified by E. Cartan.

A simply connected Riemannian symmetric space is said to be irreducible if it is not the product of two or more Riemannian symmetric spaces.

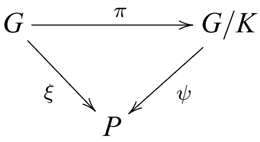
Let $G := \mathrm{GL}_n(\mathbb{C})$ and $K := \mathrm{U}(n)$, $P := \mathbb{P}_n$. Polar decomposition asserts that

$$G = PK.$$

We have three maps:

$$\begin{aligned} \pi :&\quad G \to G/K, &\quad g \mapsto gK \\ \xi :&\quad G \to P, &\quad g \mapsto gg^* \\ \psi :&\quad G/K \to P, &\quad gK \mapsto gg^* \end{aligned}$$

Clearly ψ is well-defined because of polar decomposition. We have the following commutative diagram:



Group actions on G/K and P

ψ is a bijection which can be used to identify G/K and P, i.e.,

$$gK \leftrightarrow gg^*, \quad \text{or} \quad p^{1/2}K \leftrightarrow p$$

The inverse of ψ is given by

$$\psi^{-1}(q) = q^{1/2}K, \quad q \in P.$$

- G acts naturally on G/K:

$$g \cdot K = gxK, \quad x, g \in G.$$

- This action is carried to P by ψ:

$$g \cdot q = gqg^*.$$

Symmetric space of noncompact type

Let G/K be a symmetric space of noncompact type. So G has Cartan decomposition $G = PK$ and G/K is a Riemannian manifold of nonpositive curvature.

Transfer the well-known Riemannian structure of G/K to P.

Theorem (LLT, 2014)

The unique geodesic joining any two points $p, q \in P$ has the parametrization

$$\gamma(t) = p\#_t q = p^{1/2}\left(p^{-1/2} q p^{-1/2}\right)^t p^{1/2}, \qquad 0 \le t \le 1.$$

• M. Liao, X. Liu, and T.Y. Tam, Geometric mean for symmetric spaces of noncompact type, Journal of Lie Theory, 24 (2014), 725-736.

Kostant's pre-order

The group structure of G enables us to extend the notion of log majorization to P and we denote the extended pre-order by $\prec_G$.

It will reduce to the usual log majorization if $G = \mathrm{SL}_n(\mathbb{C})$.

The definition of the pre-order is due to Kostant (1973) and can be done on two levels: Lie group level and Lie algebra level.

The definition is quite involved: Complete multiplicative Jordan decomposition, hyperbolic element, Weyl group, etc. Thus we skip the definition.

Theorem (LLT, 2014)

Let $p, q \in P$. For any $0 \le t \in 1$ and $0 < r \le 1$,

$$\begin{aligned} p\#_t q &\prec_G e^{(1-t)\log p + t\log q} \\ &\prec_G \left(p^{(1-t)r} q^{tr}\right)^{1/r} \\ &\prec_G p^{1-t} q^t. \end{aligned}$$

- The first inequality is an extension of a result of Ando and Hiai.
- The second inequality is related to the famous Golden-Thompson-Kostant inequality: $e^{H+K} \prec_{\log} e^H e^K$.
- The last inequality is an extension of a result of H. Araki, On an inequality of Lieb and Thirring, Lett. Math. Phys. 19 (1990) 167-170.

Theorem (LLT, 2014)

For any $p, q \in P$ and $0 \leq r \leq 1$,

$$(p\#_t q)^r \prec_G p^r \#_t q^r.$$

Theorem (LLT, 2014)

For any $0 \leq t \leq 1$ and $H, K \in \mathfrak{p}$,

$$\exp\{(1-t)H + tK\} = \lim_{r \to 0} \{\exp(rH)\#_t \exp(rK)\}^{1/r}$$

where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $P = \exp \mathfrak{p}$.

These are extensions of the results of Ando and Hiai, and Hiai and Petz.

Tangent vector to the geodesic

Let $\mathfrak{l}$ denote the set of all real semi-simple elements in $\mathfrak{g}$. For any two points $s, r \in G/K$, we associate a real semi-simple element $x(r,s) \in \mathfrak{l}$ as follows: Since G acts naturally on G/K, let

$$K_r := \{g \in G : g \cdot r = r\}$$

be the isotropy subgroup of G at r. Write $r = p^{1/2}K$. So it is easy to see that

$$K_r = p^{1/2}Kp^{-1/2}.$$

Now $\mathfrak{k}_r = \mathrm{Ad}\,(p^{1/2})\mathfrak{k}$ is the Lie algebra of K_r and

$$\mathfrak{g} = \mathfrak{k}_r + \mathfrak{p}_r$$

is a Cartan decomposition of $\mathfrak{g}$ in which $\mathfrak{p}_r := \mathrm{Ad}\,(p^{1/2})\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}_r$ in $\mathfrak{g}$.

The exponential map $\sigma_r : \mathfrak{p}_r \to G/K$ at r is a bijection. More precisely,

$$\sigma_r(x) := (\exp x) \cdot r = (\exp x) p^{1/2} K = p^{1/2} \exp\{\mathrm{Ad}\,(p^{-1/2})x\} K.$$

Since any two points of Riemannian symmetric space G/K can be joined by a unique geodesic, there is a unique element

$$x(r,s) \in \mathfrak{p}_r \subset \mathfrak{l}$$

such that $\sigma_r(x(r,s)) = s$. So $x(r,s)$ is the tangent vector at r to the geodesic joining r and s such that $\|x(r,s)\| = d(r,s)$.
Fact:

$$\exp x(r,s) = (qp^{-1})^{1/2},$$

where $r = p^{1/2}K$ and $s = q^{1/2}K$.

Generalization of Bhatia's inequality

Recall Bhatia's inequality: Given $A, B \in \mathbb{P}_n$,

$$d(A\#_t B, A\#_t C) \leq t d(B, C), \quad 0 \leq t \leq 1$$

Theorem (LLT, 2014)

Let $o, r, s \in G/K$. For each $0 \leq t \leq 1$,

$$x(o\#_t r, o\#_t s) \prec_G t\, x(r, s).$$

Corollary

Let $o, r, s \in G/K$. For each $0 \leq t \leq 1$,

$$d(o\#_t r, o\#_t s) \leq t d(r, s).$$

Equivalently, if $A, B, C \in P$, then $d(A\#_t B, A\#_t C) \leq t d(B, C)$.

Thank You!