Nonsymmetric normal entry patterns with the maximum number of distinct indeterminates

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Patterns of matrices

- Symmetric matrices
- Toeplitz matrices
- Hankel matrices
- Circulant matrices

General situation: entry pattern

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- Example:

$$A = \begin{bmatrix} x & y & z \\ z & x & y \\ u & z & x \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 5 \\ 5 & 2 & 3 \\ 7 & 5 & 2 \end{bmatrix} \in \mathscr{R}(A).$$

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Problem 2

Determine the maximum number of distinct indeterminates in a (nonsymmetric) normal entry pattern of a given order and the patterns that attain this number.

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Theorem 3 (H., Zhan, LAA, 2016)

Let A_i be the coefficient matrix of x_i in an entry pattern $A \in M_n\{x_1, \ldots, x_k\}$, $i = 1, \ldots, k$. Then A is a normal entry pattern if and only if each A_i is normal and

$$A_i A_j^T + A_j A_i^T = A_i^T A_j + A_j^T A_i \quad \text{for all} \quad 1 \le i < j \le k.$$

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Corollary 4 (H., Zhan, LAA, 2016)

Let A_1 be the coefficient matrix of x_1 in $A \in M_n\{x_1, x_2\}$. Then A is a normal entry pattern if and only if A_1 is normal.

Maximum number of distinct entries in normal entry patterns

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- The entry pattern of order n with maximum number of distinct entries is

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix}$$

  where x_{ij} are all distinct indeterminates for $1 \leq i \leq j \leq n$.

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Theorem 5 (H., Zhan, LAA, 2016)

Let $n \geq 3$ be an integer, and let A be a nonsymmetric normal entry pattern of order n with k distinct entries. Then $k \leq n(n-3)/2 + 3$, *where equality holds if and only if A is permutation similar to a pattern of the form*

$$\left[\begin{array}{cccc|ccc} x_{11} & x_{12} & \cdots & x_{1,n-3} & y_1 & y_1 & y_1 \\ x_{12} & x_{22} & \cdots & x_{2,n-3} & y_2 & y_2 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ x_{1,n-3} & x_{2,n-3} & \cdots & x_{n-3,n-3} & y_{n-3} & y_{n-3} & y_{n-3} \\ \hline y_1 & y_2 & \cdots & y_{n-3} & z & u & v \\ y_1 & y_2 & \cdots & y_{n-3} & v & z & u \\ y_1 & y_2 & \cdots & y_{n-3} & u & v & z \end{array}\right] \quad (1)$$

where u, v, z, y_i, x_{ij}, $1 \leq i \leq j \leq n-3$, are distinct indeterminates.

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- We use induction on the order n to prove that if A is a nonsymmetric normal entry pattern of order n, then $\phi(A) \leq n(n-3)/2 + 3$.

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  To the contrary, assume that $\phi(A) \geq n(n-3)/2 + 4$. Then A contains an entry, say, x_1, which appears exactly once or twice in A. Otherwise, each entry appears at least 3 times and we have

  $$3[n(n-3)/2 + 4] > n^2,$$

  which contradicts the fact that A has only n^2 entries.

Sketch of the proof

Let A_i be the coefficient matrix of x_i in A. If $f(A_1) = 1$, then A is permutation similar to

$$A^{(1)} = \begin{bmatrix} x_1 & a \\ a^T & B \end{bmatrix}$$

with B a normal entry pattern of order $n - 1$.

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If $f(A_1) = 2$, then A is permutation similar to

$$\begin{bmatrix} x_1 & x_j & b \\ x_j & x_1 & c \\ b^T & c^T & B \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x_j & x_1 & b \\ x_1 & x_k & c \\ b^T & c^T & B \end{bmatrix}$$

where $j \neq 1$ and $k \neq 1$.

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- Use induction on the order n to determine the nonsymmetric normal entry patterns of order n with $\phi(A) = n(n-3)/2 + 3$ distinct indeterminates.

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  From $4[n(n-3)/2 + 3] > n^2$ we know that there is at least one entry that appears less than 4 times in A. Suppose x_1 is an entry that appears the least times in A. Then $f(A_1) \leq 3$. We distinguish three cases:

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- Use induction on the order n to determine the nonsymmetric normal entry patterns of order n with $\phi(A) = n(n-3)/2 + 3$ distinct indeterminates.

  From $4[n(n-3)/2 + 3] > n^2$ we know that there is at least one entry that appears less than 4 times in A. Suppose x_1 is an entry that appears the least times in A. Then $f(A_1) \leq 3$. We distinguish three cases:

  Case 1. $f(A_1) = 1$: A is permutation similar to (1).

  Case 2. $f(A_1) = 2$: deduce a contradiction.

  Case 3. $f(A_1) = 3$: deduce a contradiction.

Open problems

- We say that an entry pattern A *requires all real eigenvalues* if the eigenvalues of any matrix in $\mathscr{R}(A)$ are all real numbers.

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Characterize those entry patterns that require all real eigenvalues.

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Characterize those entry patterns that require all real eigenvalues.

- Example:

$$\begin{bmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ x_2 & x_2 & x_2 & \cdots & x_2 \\ x_3 & x_3 & x_3 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ x_n & x_n & x_n & \cdots & x_n \end{bmatrix}.$$

Thank you

for

your attention!

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Welcome to Changsha next year!