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Application of the van der Pauw method for samples with holes

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Abstract

Modifications of the original van der Pauw relation were suggested recently. The methods are applicable to samples with a hole, unlike the original van der Pauw relation, but it takes too much time and effort to apply the methods to samples with high randomness. So in this paper we suggest two generalizations of the van der Pauw method which are applicable for two-dimensional homogeneous systems with a finite number of holes. Both methods rely on obtaining a crucial constant of the system, ν. The first method involves setting the probes far from each other while conducting the experiment using a sample with a small hole, approximating a relation that gives ν as a linear function of the area ratio of the hole only. The second method produces an identical sample of known resistivity and thickness to obtain ν, which is believed to be dependent on the geometrical properties only. Unlike the earlier methods, which entailed complex procedures, little in the way of measurements and computation is needed for the new methods. The methods will be very useful in electric experiments or industries which need to measure the resistivity of samples.

Keywords: van der Pauw method, conformal mapping, resistivity measurement, defects

(Some figures may appear in colour only in the online journal)

1. Introduction

In 1958 van der Pauw showed that the sheet resistivity of a metallic sample can be obtained from any arrangement of four point contacts on the edge of a flat homogeneous sample of known thickness [1]. The method uses two probes to supply current to the sample, and another two probes to measure the electric potential difference. Van der Pauw defined the resistance $R_{PQ,RS}$ as $R_{PQ,RS} = (V_S - V_R)/j_{PQ}$ using the potential difference between the contact points R and S when current j_{PQ} flows from P to Q. $R_{QR,SP}$ is defined in the same way. The original van der Pauw relation is given as:

$$\exp(-R_{PQ,RS}/\lambda) + \exp(-R_{QR,SP}/\lambda) = 1 \qquad (1)$$

where $\lambda = \rho/\pi d$ and ρ is the resistivity of the sample with thickness d. The van der Pauw relation has important practical applications in measuring the resistivity and Hall coefficient for samples of arbitrary shape [1], and has been applied to different types of transport phenomena such as heat transport measurements [2, 3]. However, the van der Pauw relation (equation (1)) is only applicable to homogeneous samples with uniform thickness and no holes. So when we use the van der Pauw method for samples with defects (samples with holes, non-homogeneous samples, etc) we cannot measure the exact value of the resistivity of the given sample.

So in order to use the van der Pauw method to measure the resistivity of samples with defects, scientists have developed a method of calculating the measuring error, which is proportional to the sensitivity of the sheet resistance (resistivity divided by the sample thickness) if the size of the hole is small enough [4–10]. The sensitivity of the sheet resistance depends on the sample and the geometry of its hole. Although we are

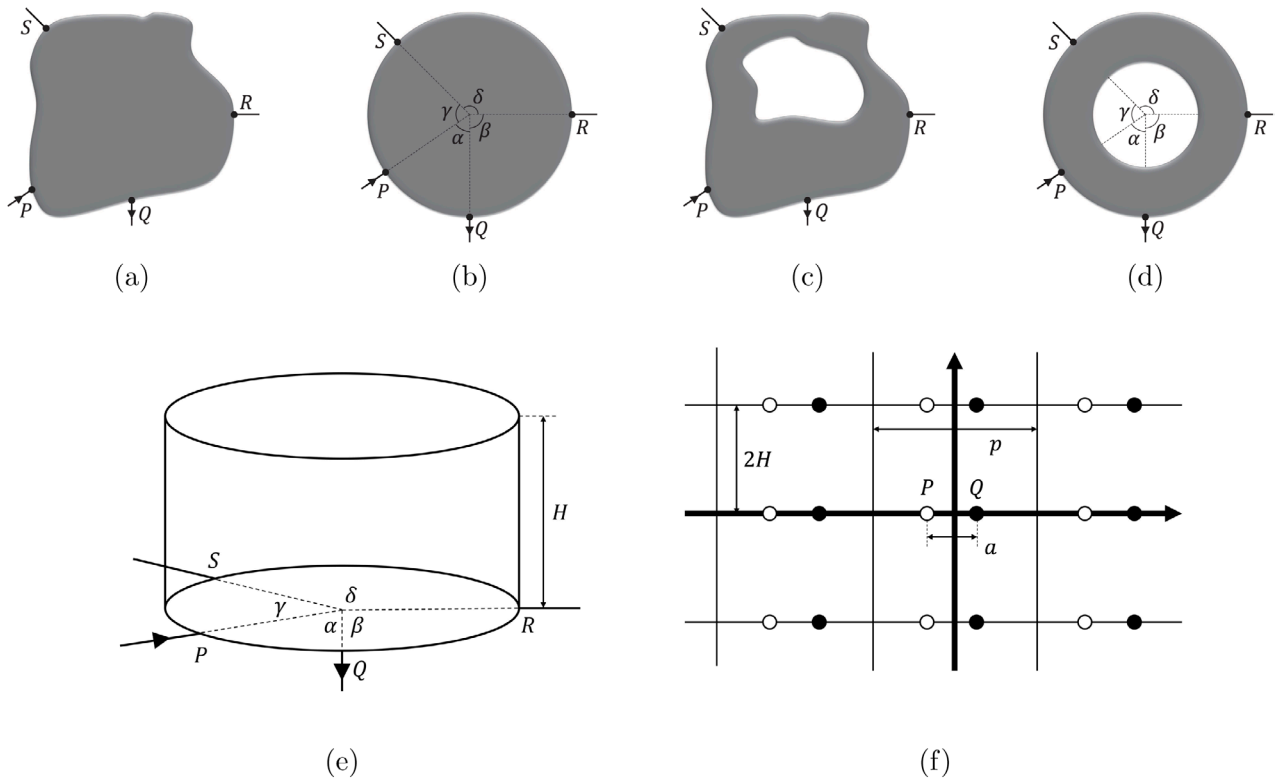


Figure 1. Sample (a) can be conformally mapped to a circle (b). Similarly a sample with a hole (c) can be conformally mapped to an annulus (d) or a cylinder (e) of height H and circumference p. A cylinder can also be identified as a periodic lattice (f). The Riemann modulus of the sample is defined on the cylinder or lattice as $h = 4\pi H/p$, and on the annulus as $h = 2\ln R/r$, but there is no algebraic expression for a arbitrary sample with a hole.

able to calculate the resistivity of a given sample with a hole if we know the sensitivity of the sample, calculation of the sensitivity is complicated, and the process of computing the sensitivity has to be repeated every time the sample's geometry is changed. Research into the influence of certain types of cracks and holes in the sample has been done [11]. However, this research can only be applied to limited types of sample defects.

Recently, work has been done to modify the original van der Pauw method and develop a method of finding out the value of ν, which is crucial for calculating the sheet resistivity of the sample [12, 13]. We call this the modified van der Pauw method. However, the method is very complicated and is only applicable to very limited types of sample geometry. Moreover, we are not able to calculate the resistivity of the sample if it has multiple holes.

A six-probe generalization has been proposed to make the modified van der Pauw method applicable to all types of sample geometry [14]. Using this method, there are seven unknowns (five angular variables, the Riemann modulus and sheet resistivity), but nine independent modified van der Pauw relations can be given, allowing the determination of all unknown variables. However, this method is highly computationally inefficient, and with heuristics being used to solve these highly nonlinear equations the error can be significantly large; therefore this method is generally inapplicable. So the six-probe generalization was unable to overcome the problems with the modified van der Pauw method. Currently, the modified van der Pauw method and its six-probe generalization are the only modifications which can be applied to samples with single hole.

The present work was done in order to improve the modified van der Pauw method, which has limited applicability, and derive a universal method for samples with various geometries, including samples with multiple holes. Also, the new method should be computationally efficient and convenient to use.

2. A brief review of the modified van der Pauw method

It has recently been investigated theoretically and partially experimentally that in the case of a sample with one isolated hole the original van der Pauw relation becomes an inequality [12]:

$$\exp(-R_{PQ,RS}/\lambda) + \exp(-R_{QR,SP}/\lambda) = \nu \leqslant 1. \qquad (2)$$

The original van der Pauw method can be proved by conformally mapping an arbitrary sample to a circle, that can also be identified as a periodic line of points. A similar idea was used to map a sample with a hole to a cylinder of finite length, that can also be identified as a periodic lattice (figure 1). Calculating the potential in the periodic lattice, the four probe resistivities are given as a function of the sheet resistivity, h, the Riemann modulus and $\alpha, \beta, \gamma, \delta$, the angles between probes after they have been mapped:

$$R_{PQ,RS} = \lambda \ln \left| \frac{G(\alpha+\beta,h)G(\alpha+\gamma,h)}{G(\beta,h)G(\gamma,h)} \right|,$$

$$R_{QR,SP} = \lambda \ln \left| \frac{G(\alpha+\beta,h)G(\alpha+\gamma,h)}{G(\alpha,h)G(\delta,h)} \right|,$$

$$R_{PR,QS} = \lambda \ln \left| \frac{G(\alpha,h)G(\delta,h)}{G(\beta,h)G(\gamma,h)} \right|,$$

$$G(\theta,h) = \sin\frac{\theta}{2}\prod_{n=1}^{\infty}\left(1-\frac{\cos\theta}{\cosh hn}\right). \quad (3)$$

Using these relations the inequality can be proven as:

$$\nu = \frac{G(\alpha,h)G(\delta,h)+G(\beta,h)G(\gamma,h)}{G(\alpha+\beta,h)G(\alpha+\gamma,h)} \leqslant 1. \quad (4)$$

While it is known that any sample with a hole can be conformally mapped to a cylinder, the exact numerical relation that provides mapping is only known for certain geometries such as a rectangular hole in a rectangular sample with identical centers, or elliptically shaped samples and holes [15]. As such, in general geometries, the angles between probes after they have been mapped and the Riemann modulus h of the sample is not known, resulting in too many unknowns for the traditional van der Pauw method to be applicable.

In this paper we present two methods to calculate the value ν. The first method uses the fact that the value of ν is not very sensitive to small changes in the values of $\alpha, \beta, \gamma, \delta$. So we made an approximation of ν which is very easy to calculate. We will call this method 1. The second method uses the fact that the ν value of the given sample depends only on the geometry, not the type of material of which the sample is made. We will call this method 2. Both methods are derived theoretically then confirmed experimentally.

3. Methodology

We used aluminum sheets (99.99% purity) of thickness 40 μm and a size of 10 cm by 10 cm with one or more holes for experiments on method 1, and both aluminum sheets and copper sheets (99.99% purity) of thickness 40 μm and size 10 cm by 10 cm with one or more holes for experiments on method 2. The metal sheets we used in the experiments are homogeneous and have a uniform thickness. The resistivity of the aluminum sheets is $3.03 \times 10^{-8}\,\Omega \cdot$ m, and the resistivity of the copper sheets is $1.84 \times 10^{-8}\,\Omega \cdot$ m. The resistivity values of the metal sheets were obtained by applying the original van der Pauw method to sheets without any holes. We used a razor blade to make holes in the metal sheets. The area of the holes was measured by edge detection and image processing of photographs of the samples with the C++ program. The current was supplied to the samples using a DC power supply (Good Will Instrument, model GPS-4303); the value of the current was 3.22 A for methods 1 and 2. During the experiment, we used an infrared thermal camera (FLIR model E40) to check the temperature change of the metallic sample, and a temperature change of less than 1 K was observed. This is because the measurement of the potential difference between the two probes takes no more than a few seconds. Therefore the rate of change of resistivity values of the metal sheets is smaller than 0.5%, so the warming-up effect of the samples could be excluded. We used an ammeter (Hwa Shin Electronic Works, model HS-7017AM, analog type, 0.1 A resolution) and a voltmeter (Hwa Shin Electronic Works, model HS-0001, analog type, 0.1 mV resolution) to measure the current and potential difference, respectively. The power supply, ammeter and the sample were connected in a series circuit using copper wires. The voltmeter was connected to the sample in parallel. The numerical calculations required to solve the van der Pauw relation for the value of ν or the resistivity of the sample were performed using a C++ implementation of the Newton–Raphson method.

4. Results for method 1

4.1. Theoretical approach

It can be observed that when the probes are placed far apart from each other while conducting the experiment, and when the hole in the sample is small enough, it is likely that the condition $\alpha = \beta = \gamma = \delta = \pi/2$ is met. Numerical calculation of ν according to equation (4) shows that for cases where a displacement ϕ is applied as $\alpha \leftarrow \pi/2 + \phi, \beta \leftarrow \pi/2 - \phi$ or $\alpha \leftarrow \pi/2 + \phi, \delta \leftarrow \pi/2 - \phi$, the value of ν does not change significantly (figure 2). This means that value of ν is not sensitive to small changes to the values of $\alpha, \beta, \gamma, \delta$, which are the angles between the adjacent probe points after conformally mapping the sample into a cylinder. So by simply placing the probes to maximize the distance between points when conducting the experiment, the condition $\alpha = \beta = \gamma = \delta = \pi/2$ can be used without too much error.

However, the approximation of $\alpha, \beta, \gamma, \delta$ values could be only used when the value of h, which satisfies the relation $e^{-h} = A_{\text{hole}}/A$, is large. The relation can be derived from the fact that the variable h is defined as $2 \ln R/r$ for the annulus while conformally mapping the given sample, and the area ratio of the given sample's hole is approximately equal to the area ratio of the annulus when the hole is small enough [16, 17]. The value of ν cannot be bigger than 1 no matter what the values of $\alpha, \beta, \gamma, \delta$ are, so the maximum value of ν is always 1. If we decrease the value of h, the minimum value of ν (according to equation (4), the ν value is smallest when $\alpha = \beta = \gamma = \delta = \pi/2$) will also decrease. This means that ν will be more sensitive to changes in $\alpha, \beta, \gamma, \delta$ values when h is smaller. So the approximation of $\alpha = \beta = \gamma = \delta = \pi/2$ can only be used when h is big and the hole is small enough.

Setting all angles as $\pi/2$ for equation (4) and assuming that the value of h is big enough (which means that the hole is small enough) gives ν as:

$$\nu = \prod_{n=1}^{\infty}\left(1+\frac{1}{\cosh hn}\right)^{-2}. \quad (5)$$

Taking the first-order approximation of equation (5),

$$\nu \sim 1 - 4e^{-h}. \quad (6)$$

The linear approximation above should stand for samples with a single hole.

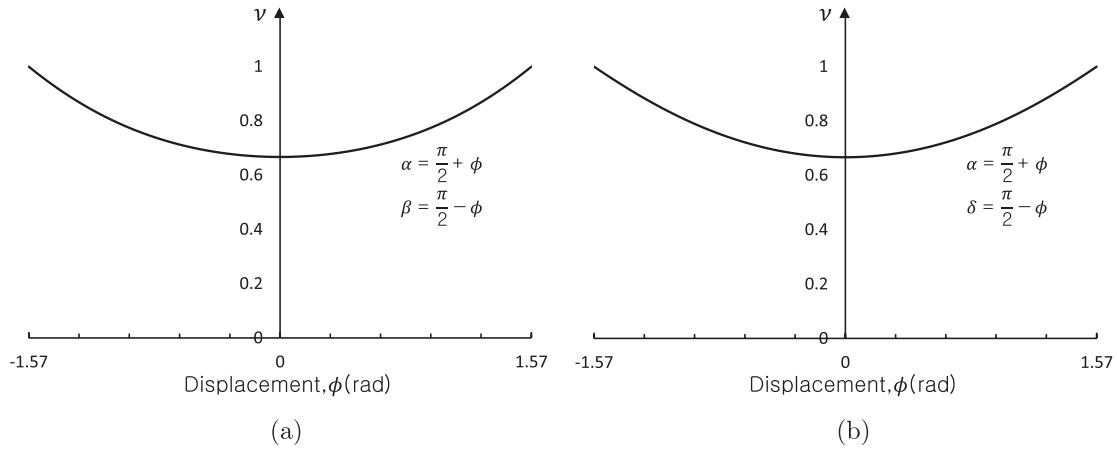


Figure 2. In the region of $\phi = 0$ small changes to ϕ do not result in significant changes to ν. In the graph, $h = 2.303$, which means that $A_{\text{hole}}/A = 0.1$. The graph was drawn using C++ coding. The value of ν was calculated for every 0.01 rad, from -1.57 rad to 1.57 rad. (a) Displacement to α and β. (b) Displacement to α and δ.

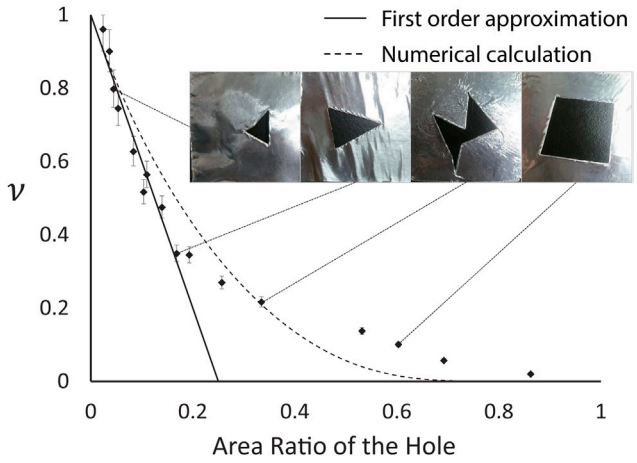


Figure 3. Experimental data for the ν values of samples with a single hole are compared with the numerical calculation of equation (5) and the first-order approximation in the form of equation (6). The distances between the probes were maximized during the experiment in order to use the approximation $\alpha = \beta = \gamma = \delta = \pi/2$ (equations (5) and (6) both use the approximation). The area ratio of the hole was found using edge detection and image processing. We can see that equations (5) and (6) both provide substantial approximations to the value ν.

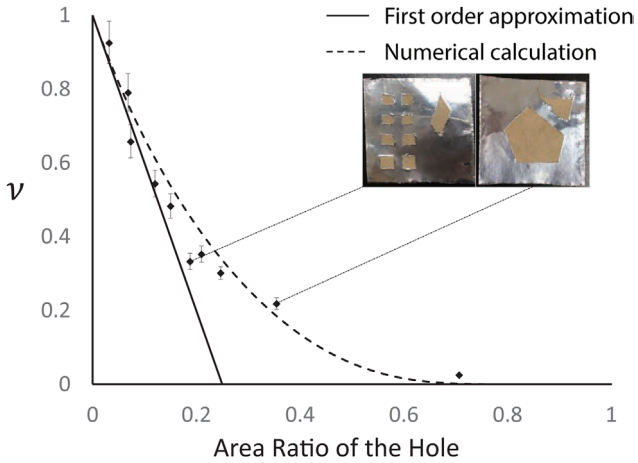


Figure 4. Samples with multiple holes followed the linear approximation of equation (6).

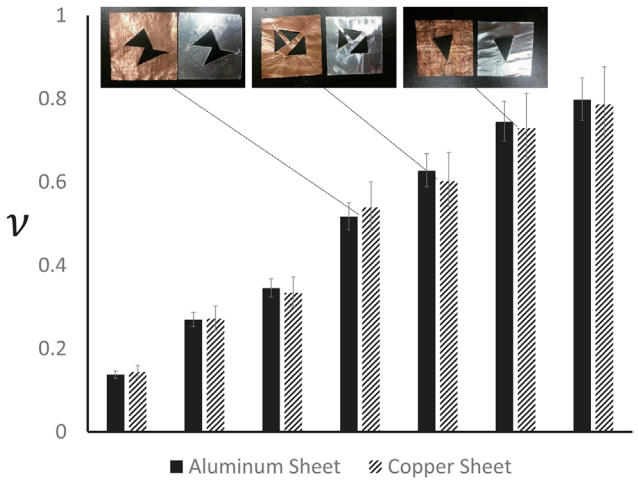


Figure 5. A comparison of the values of ν obtained from aluminum and copper sheets for nearly identical samples with nearly identical probe locations shows that the differences are within the error margins of measurement. The values showed an average error of 1.42%, always smaller than the error of measurement.

4.2. Experimental results

The linear approximation done above is only valid if h is large. We recommend using the linear approximation only when the area ratio of the holes on the sample is smaller than 0.2, which makes the value of h big enough. Figure 3 shows that the experimental data fit well to the approximation when the area ratio of the hole is smaller than 0.2($R^2 = 0.971$). While numerical calculation of equation (5) provides higher accuracy throughout the whole range of area ratio values, the linear approximation is more computationally efficient.

Also, when we used samples with more than one hole for the experiment (figure 4), the ν values of the samples followed the same linear approximation (equation (6)) even though the approximation was derived for samples with a single hole.

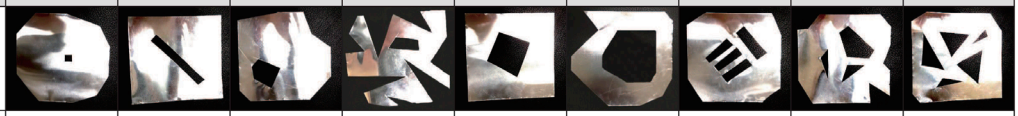
Sample no.	#1	#2	#3	#4	#5	#6	#7	#8	#9
Sample Picture									
$\nu_{method\ 1}$	0.976	0.748	0.731	0.684	0.403	-	-	-	-
$\rho_{method\ 1}$ ($\times 10^{-8}$ $\Omega \cdot m$)	2.98	2.97	3.00	2.89	2.92	-	-	-	-
Error rate method 1 (%)	-1.65	-1.98	-0.99	-4.62	-3.63	-	-	-	-
$\nu_{method\ 2}$	0.985	0.767	0.738	0.715	0.425	0.238	0.503	0.513	0.368
$\rho_{method\ 2}$ ($\times 10^{-8}$ $\Omega \cdot m$)	3.02	3.05	3.03	3.01	3.02	3.00	3.04	3.01	3.00
Error rate method 2 (%)	-0.33	+0.66	0.00	-0.66	-0.33	-0.99	+0.33	-0.66	-0.99

Figure 6. Experimental resistivity values of the nine given samples obtained with methods 1 and 2. Method 1 was used to obtain the resistivity values of the samples with a single hole, and method 2 was used for all samples. The error rate of both methods was small enough for the actual use of the methods.

However, using the approximation to samples with multiple holes is not recommended because there is no theoretical proof that the approximation is applicable for samples with multiple holes.

5. Results for method 2

5.1. Theoretical approach

One can observe that the value of ν is a constant 1 for a sample without a hole, and for a sample with an isolated hole the value is dependent only on the topological parameters of the sample. Thus for two samples with different material properties but identical geometries and probe points, ν will be the same. We investigate whether this holds for all geometries, including samples with multiple holes.

The theoretical proof of this is trivial. Samples with the same geometry but made of different materials have the same boundary conditions. If we supply an identical amount of current to the samples with the same geometry, the normal derivative of the electric potential function at the two current electrodes would be simply scaled by a factor of the given sample's sheet resistance, since the electric potential function could be seen as a boundary condition problem. So the potentials should be scaled by the same factor (assuming an identical current density), and the values of $R_{PQ,RS}$ and $R_{QR,SP}$ would also be scaled by the value of sheet resistance. Thus, by equation (2), the ν values of samples with the same geometry are equal. This stands even when the samples have multiple holes.

Also, the eligibility of method 2 can be proven by weighting and sensitivity functions.

5.2. Experimental results

Comparison between copper and aluminum sheets shows that the value of ν is similar for both, with differences falling within the error rate (figure 5). Most of the error is believed to be a result of errors in the measurement process and slight deviations in the location of probes—errors that can be improved in a more advanced experimental setting. By scanning a sample with an unknown resistivity and making an identical (the thickness of the two samples could be different, but must be known) sample of known resistivity, we can measure the value of ν and use this value to apply the modified van der Pauw relation to the unknown sample, thus obtaining the sheet resistivity of the original sample.

6. Usage of methods 1 and 2

To verify the two methods, we used the two methods that we have proposed to obtain the resistivity of random samples made with 0.1 mm thick aluminum sheets (figure 6). The experimental setup introduced earlier was used to conduct the experiment. We used the linear approximation (equation (6)) to calculate the resistivity of the given samples with method 1. For method 2, we made another sample which had the same geometry as the given sample, using copper sheets. We used the copper sample to obtain the ν value of the given sample. The results using both methods 1 and 2 are shown in the table in figure 6. For the samples with a small single hole ($A_{\text{hole}}/A < 0.2$, samples 1–5), methods 1 and 2 were both used; method 2 was used for samples with a big single hole ($A_{\text{hole}}/A > 0.2$, sample 6) and samples with multiple holes (samples 7–9).

The results experimentally confirm the two methods proposed in this paper. The second method shows very small error rates for samples with one or more holes, and the first method shows acceptable error rates for samples with a single hole. The error rates of the first method always have a negative value, which is because the value of ν is a minimum when $\alpha = \beta = \gamma = \delta = \pi/2$ [12]. The error rates of the second method were not biased. While use of the second method will allow researchers to obtain more accurate resistivity values of samples with multiple holes, use of the first method for samples with a single hole takes much less time and work.

7. Conclusion and discussion

In this research, we generalized the van der Pauw method for samples of various geometries and proposed two methods for obtaining ν, which is a crucial factor for calculating the sheet resistivity of a sample. The first method involved setting the probes far from each other while conducting the experiment using a sample with a small hole, approximating a relation that gives ν as a linear function of the area ratio of the hole only. The second method involved the use of an identical sample of known specific resistivity and thickness to obtain ν, which is believed to be dependent on geometrical properties only. Both methods have been confirmed experimentally.

The novelty of the work is in the methods of approximating the value of ν of a given sample with unknown sheet resistivity. Using the first method we can approximate ν for a sample with a single hole using a linear relation of the area ratio of the hole. The approximation stands regardless sample's geometry if the area of the hole is less than 20% of the area of the sample. Unlike the existing methods, which require very complex procedures, the new method needs very little computation. The only factor we need to know is the area ratio of the hole, and this could be easily obtained by many means such as image processing a photograph of the given sample. Also, the first method could be very useful in electronics industries such as semiconductor production. Engineers often need to measure the sheet resistivity of various samples with pinholes, scratches or holes for the circuits when producing semiconductor parts. In this case, the hole in the sample would be small enough, so instead of dividing the sample into parts and applying the traditional van der Pauw relation, we can just use the approximation of method 1. This will save a lot of time and effort, but the acquired data will still be accurate. And if the holes on the sample are too big to use the first method (which is not very likely when conducting real-life experiments), or if there are multiple holes in the sample, method 2 can be used instead. We believe that these methods will increase the efficiency of the measurements of sheet resistivity of various samples.

We observed that the ν values of the samples with more than one hole also follow a linear approximation (equation (6)) during experiments using method 1 (figure 4). We are currently working to find a theoretical proof that the approximation is applicable to samples with multiple holes.

References

[1] van der Pauw L J 1958 A method of measuring specific resistivity and Hall effect of discs of arbitrary shape *Phil. Res. Repts* **13** 1–9

[2] de Boor J and Schmidt V 2010 Complete characterization of thermoelectric materials by a combined van der Pauw approach *Adv. Mater.* **22** 4303–7

[3] de Boor J and Schmidt V 2011 Efficient thermoelectric van der Pauw measurements *Appl. Phys. Lett.* **99** 022102

[4] Koon D W 1989 Effect of contact size and placement, and of resistive inhomogeneties on van der Pauw measurements *Rev. Sci. Instrum.* **60** 271–4

[5] Koon D W, Bahl A A and Duncan E O 1989 Measurement of contact placement errors in the van der Pauw technique *Rev. Sci. Instrum.* **60** 275–6

[6] Koon D W and Knickerbocker C J 1992 What do you measure when you measure resistivity? *Rev. Sci. Instrum.* **63** 207–10

[7] Koon D W and Knickerbocker C J 1997 Effects of macroscopic inhomogeneities on resistive and Hall measurements on crosses, cloverleafs, and bars *Rev. Sci. Instrum.* **67** 4282–5

[8] Koon D W and Chan W K 1998 Direct measurement of the resistivity weighting function *Rev. Sci. Instrum.* **69** 4218–20

[9] Koon D W, Wang F, Peterson D H and Hansen O 2014 Sensitivity of resistive and Hall measurements to local inhomogeneities: finite-field, intensity, and area corrections *J. Appl. Phys.* **116** 133706

[10] Koon D W, Heřmanová M and Náhlík J 2015 Electrical conductance sensitivity functions for square and circular cloverleaf van der Pauw geometries *Meas. Sci. Technol.* **26** 115004

[11] Náhlík J, Kašpárková I and Fitl P 2011 Study of quantitative influence of sample defects on measurements of resistivity of thin films using van der Pauw method *Measurement* **44** 1968–79

[12] Szymański K, Cieśliński J L and Lapiński K 2013 Van der Pauw method on a sample with an isolated hole *Phys. Lett.* A **377** 651–4

[13] Szymański K, Cieśliński J L and Lapiński K 2015 Determination of the Riemann modulus and sheet resistance of a sample with a hole by the van der Pauw method *Meas. Sci. Technol.* **26** 055003

[14] Szymański K, Lapiński K, Cieśliński J L, Kobus A, Zaleski P, Biernacka M and Perzyńska K 2015 Determination of the Riemann modulus and sheet resistivity by a six-point generalization of the van der Pauw method *Meas. Sci. Technol.* **26** 085012

[15] Ahlfors L 1979 *Complex Analysis* 3rd edn (New York: McGraw-Hill) pp 255–7

[16] Burbea J 1981 The modular function and the modulus of a doubly-connected region *Rocky Mt. J. Math.* **11** 407–18

[17] Burbea J 1978 Effective methods of determining the modulus of doubly connected domains *J. Math. Anal. Appl.* **62** 236–42