Algebraic generating functions for languages avoiding Riordan patterns

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Outline

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Definition in terms of $d(t)$ and $h(t)$

- A *Riordan array* is a pair

$$D = \mathcal{R}(d(t),\ h(t))$$

  in which $d(t)$ and $h(t)$ are formal power series such that $d(0) \neq 0$ and $h(0) = 0$
- if $h'(0) \neq 0$ the Riordan array is called *proper*
- it denotes an infinite, lower triangular array $(d_{n,k})_{n,k \in N}$ where:

$$d_{n,k} = [t^n]d(t)h(t)^k$$

The A and Z sequences

An alternative definition, is in terms of the so-called A-sequence and Z-sequence, with generating functions $A(t)$ and $Z(t)$ satisfying the relations:

$$h(t) = tA(h(t)), \quad d(t) = \frac{d_0}{1 - tZ(h(t))} \quad \text{with} \quad d_0 = d(0).$$

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots$$

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots$$

The A-matrix [MSRV97]

$$d_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j}d_{n-i,k+j} + \sum_{j\geq 0}\rho_j d_{n+1,k+j+2}$$

Matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}}$ is called the A-matrix of the Riordan array. If, for $i \geq 0$:

$$P^{[i]}(t) = \alpha_{i,0} + \alpha_{i,1}t + \alpha_{i,2}t^2 + \alpha_{i,3}t^3 + \dots$$

and $Q(t)$ is the generating function for the sequence $(\rho_j)_{j\in\mathbb{N}}$, then we have:

$$\frac{h(t)}{t} = \sum_{i\geq 0} t^i P^{[i]}(h(t)) + \frac{h(t)^2}{t}Q(h(t))$$

$$A(t) = \sum_{i\geq 0} t^i A(t)^{-i}P^{[i]}(t) + tA(t)Q(t)$$

Outline

Binary words avoiding a pattern

- We consider the language $\mathcal{L}^{[\mathfrak{p}]}$ of binary words with no occurrence of a pattern $\mathfrak{p} = p_0 \cdots p_{h-1}$
- The problem of determining the generating function counting the number of words *with respect to their length* has been studied by several authors:

1. L. J. Guibas and M. Odlyzko. Long repetitive patterns in random sequences. *Zeitschrift für Wahrscheinlichkeitstheorie*, 53:241–262, 1980.
2. R. Sedgewick and P. Flajolet. *An Introduction to the Analysis of Algorithms*. Addison-Wesley, Reading, MA, 1996.

- The fundamental notion is that of the *autocorrelation vector* of bits $c = (c_0, \ldots, c_{h-1})$ associated to a given $\mathfrak{p}$

The pattern $\mathfrak{p} = 10101$

1	0	1	0	1	Tails				c_i
1	0	1	0	1					1
	1	0	1	0	1				0
		1	0	1	0	1			1
			1	0	1	0	1		0
				1	0	1	0	1	1

The autocorrelation vector is then $c = (1, 0, 1, 0, 1)$ and $C^{[\mathfrak{p}]}(t) = 1 + t^2 + t^4$ is the associated autocorrelation polynomial

Count respect bits 1 and 0

The gf counting the number F_n of binary words with length n not containing the pattern $\mathfrak{p}$ is

$$F(t) = \frac{C^{[\mathfrak{p}]}(t)}{t^h + (1-2t)C^{[\mathfrak{p}]}(t)}$$

Taking into account the number of bits 1 and 0 in $\mathfrak{p}$:

$$F^{[\mathfrak{p}]}(x,y) = \frac{C^{[\mathfrak{p}]}(x,y)}{x^{n_1^{[\mathfrak{p}]}} y^{n_0^{[\mathfrak{p}]}} + (1-x-y)C^{[\mathfrak{p}]}(x,y)}$$

where $h = n_0^{[\mathfrak{p}]} + n_1^{[\mathfrak{p}]}$ and $C^{[\mathfrak{p}]}(x,y)$ is the bivariate autocorrelation polynomial. Moreover, $F_{n,k}^{[\mathfrak{p}]} = [x^n y^k]F^{[\mathfrak{p}]}(x,y)$ denotes the number of binary words avoiding the pattern $\mathfrak{p}$ with n bits 1 and k bits 0

An example with $\mathfrak{p} = 10101$

Since $C^{[\mathfrak{p}]}(x, y) = 1 + xy + x^2y^2$ we have:

$$F^{[\mathfrak{p}]}(x, y) = \frac{1 + xy + x^2y^2}{(1 - x - y)(1 + xy + x^2y^2) + x^3y^2}.$$

n/k	0	1	2	3	4	5	6	7
0	**1**	1	1	1	1	1	1	1
1	1	**2**	3	4	5	6	7	8
2	1	3	**6**	10	15	21	28	36
3	1	4	9	**18**	32	52	79	114
4	1	5	13	30	**60**	109	184	293
5	1	6	18	46	102	**204**	377	654
6	1	7	24	67	163	354	**708**	1324
7	1	8	31	94	248	580	1245	**2490**

...the lower and upper triangular parts

n/k	0	1	2	3	4	5
0	**1**					
1	**2**	**1**				
2	**6**	3	**1**			
3	**18**	9	4	**1**		
4	**60**	30	13	5	**1**	
5	**204**	102	46	18	6	**1**

$(n,k) \mapsto (n, n-k)$ if $k \leq n$

n/k	0	1	2	3	4	5
0	**1**					
1	**2**	**1**				
2	**6**	3	**1**			
3	**18**	10	4	**1**		
4	**60**	32	15	5	**1**	
5	**204**	109	52	21	6	**1**

$(n,k) \mapsto (k, k-n)$ if $n \leq k$

Matrices $R^{[\mathfrak{p}]}$ and $R^{[\bar{\mathfrak{p}}]}$

- Let $R^{[\mathfrak{p}]}_{n,k} = F^{[\mathfrak{p}]}_{n,n-k}$ with $k \leq n$. In other words, $R^{[\mathfrak{p}]}_{n,k}$ counts the number of words avoiding $\mathfrak{p}$ with n bits 1 and $n-k$ bits 0
- Let $\bar{\mathfrak{p}} = \bar{p}_0 \ldots \bar{p}_{h-1}$ be the $\mathfrak{p}$'s conjugate, where $\bar{p}_i = 1 - p_i$
- We obviously have $R^{[\bar{\mathfrak{p}}]}_{n,k} = F^{[\bar{\mathfrak{p}}]}_{n,n-k} = F^{[\mathfrak{p}]}_{k,k-n}$. Therefore, the matrices $R^{[\mathfrak{p}]}$ and $R^{[\bar{\mathfrak{p}}]}$ represent the lower and upper triangular part of the array $F^{[\mathfrak{p}]}$, respectively

Outline

Riordan patterns [MS11]

- When matrices $R^{[\mathfrak{p}]}$ and $R^{[\bar{\mathfrak{p}}]}$ are (both) Riordan arrays?
- We say that $\mathfrak{p} = p_0 ... p_{h-1}$ is a Riordan pattern if and only if

$$C^{[\mathfrak{p}]}(x, y) = C^{[\mathfrak{p}]}(y, x) = \sum_{i=0}^{\lfloor (h-1)/2 \rfloor} c_{2i} x^i y^i$$

provided that $\left| n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]} \right| \in \{0, 1\}$

1. D. Merlini and R. Sprugnoli. Algebraic aspects of some Riordan arrays related to binary words avoiding a pattern. *Theoretical Computer Science*, 412 (27), 2988-3001, 2011.

Theorem 1

Matrices

$$R^{[\mathfrak{p}]} = (d^{[\mathfrak{p}]}(t), h^{[\mathfrak{p}]}(t)), \quad R^{[\bar{\mathfrak{p}}]} = (d^{[\bar{\mathfrak{p}}]}(t), h^{[\bar{\mathfrak{p}}]}(t))$$

are both RAs $\leftrightarrow$ $\mathfrak{p}$ is a Riordan pattern.

By specializing this result to the cases $\left| n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]} \right| \in \{0, 1\}$ and by setting $C^{[\mathfrak{p}]}(t) = C^{[\mathfrak{p}]}\left(\sqrt{t}, \sqrt{t}\right) = \sum_{i \geq 0} c_{2i} t^i$, we have

Theorem 1: the case $n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]} = 1$

$$d^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t)}{\sqrt{C^{[\mathfrak{p}]}(t)^2 - 4tC^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_0^{\mathfrak{p}}})}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t) - \sqrt{C^{[\mathfrak{p}]}(t)^2 - 4tC^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_0^{\mathfrak{p}}})}}{2C^{[\mathfrak{p}]}(t)}.$$

Theorem 1: the case $n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]} = 0$

$$d^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t)}{\sqrt{(C^{[\mathfrak{p}]}(t) + t^{n_0^{\mathfrak{p}}})^2 - 4tC^{[\mathfrak{p}]}(t)^2}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t) + t^{n_0^{\mathfrak{p}}} - \sqrt{(C^{[\mathfrak{p}]}(t) + t^{n_0^{\mathfrak{p}}})^2 - 4tC^{[\mathfrak{p}]}(t)^2}}{2C^{[\mathfrak{p}]}(t)}.$$

Theorem 1: the case $n_0^{[\mathfrak{p}]} - n_1^{[\mathfrak{p}]} = 1$

$$d^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t)}{\sqrt{C^{[\mathfrak{p}]}(t)^2 - 4tC^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_1^{\mathfrak{p}}})}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t) - \sqrt{C^{[\mathfrak{p}]}(t)^2 - 4tC^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_1^{\mathfrak{p}}})}}{2(C^{[\mathfrak{p}]}(t) - t^{n_1^{\mathfrak{p}}})}.$$

Formulae for classes of patterns

- $\mathfrak{p} = 1^{j+1}0^{j}$

$$d^{[\mathfrak{p}]}(t) = \frac{1}{\sqrt{1-4t+4t^{j+1}}}, \quad h^{[\mathfrak{p}]}(t) = \frac{1-\sqrt{1-4t+4t^{j+1}}}{2}$$

- $\mathfrak{p} = 0^{j+1}1^{j}$

$$d^{[\mathfrak{p}]}(t) = \frac{1}{\sqrt{1-4t+4t^{j+1}}}, \quad h^{[\mathfrak{p}]}(t) = \frac{1-\sqrt{1-4t+4t^{j+1}}}{2(1-t^{j})}$$

- $\mathfrak{p} = 1^{j}0^{j}$ and $\mathfrak{p} = 0^{j}1^{j}$

$$d^{[\mathfrak{p}]}(t) = \frac{1}{\sqrt{1-4t+2t^{j}+t^{2j}}}, \quad h^{[\mathfrak{p}]}(t) = \frac{1+t^{j}-\sqrt{1-4t+2t^{j}+t^{2j}}}{2}$$

Formulae for classes of patterns

- $\mathfrak{p} = (10)^j 1$

$$d^{[\mathfrak{p}]}(t) = \frac{\sum_{i=0}^{j} t^i}{\sqrt{1 - 2\sum_{i=1}^{j} t^i - 3\left(\sum_{i=1}^{j} t^i\right)^2}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{\sum_{i=0}^{j} t^i - \sqrt{1 - 2\sum_{i=1}^{j} t^i - 3\left(\sum_{i=1}^{j} t^i\right)^2}}{2\sum_{i=0}^{j} t^i}$$

- $\mathfrak{p} = (01)^j 0$

$$d^{[\mathfrak{p}]}(t) = \frac{\sum_{i=0}^{j} t^i}{\sqrt{1 - 2\sum_{i=1}^{j} t^i - 3\left(\sum_{i=1}^{j} t^i\right)^2}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{\sum_{i=0}^{j} t^i - \sqrt{1 - 2\sum_{i=1}^{j} t^i - 3\left(\sum_{i=1}^{j} t^i\right)^2}}{2\sum_{i=0}^{j-1} t^i}$$

A combinatorial interpretation for $\mathfrak{p} = 10$

In this case we get the RA $\mathcal{R}^{[10]} = \left(d^{[10]}(t), h^{[10]}(t)\right)$ such that

$$d^{[10]}(t) = \frac{1}{1-t} \quad \text{and} \quad h^{[10]}(t) = t,$$

so the number $R^{[10]}_{n,0}$ of words containing n bits 1 and n bits 0, avoiding pattern $\mathfrak{p} = 10$, is $[t^n]d^{[10]}(t) = 1$ for $n \in \mathbb{N}$.
In terms of lattice paths this corresponds to the fact that there is exactly one *valley*-shaped path having n steps of both kinds $\diagup$ and $\diagdown$, avoiding $\mathfrak{p} = 10$ and terminating at coordinate $(2n, 0)$ for each $n \in \mathbb{N}$, formally the path 0^n1^n.

Outline

The $|w|_0 \leq |w|_1$ constraint

- let $|w|_i$ be the number of bits i in word w
- enumeration of binary words avoiding a pattern $\mathfrak{p}$, without the constraint $|w|_0 \leq |w|_1$, gives a rational bivariate generating function for the sequence $F_n^{[\mathfrak{p}]} = \sum_{k=0}^{n} F_{n,k}^{[\mathfrak{p}]}$
- under the restriction such that words have to have no more bits 0 than bits 1, then the language is no longer regular and its enumeration becomes more difficult
- using gf $R^{[\mathfrak{p}]}(x, y)$ and the fundamental theorem of RAs:

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n] d(t) f(h(t))$$

we obtain many **new algebraic generating functions** expressed in terms of the autocorrelation polynomial of $\mathfrak{p}$

Theorem 2: the case $n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]} = 1$

Recall that

$$R^{[\mathfrak{p}]}(t, w) = \sum_{n,k \in \mathbb{N}} R_{n,k}^{[\mathfrak{p}]} t^n w^k = \frac{d^{[\mathfrak{p}]}(t)}{1 - w h^{[\mathfrak{p}]}(t)}$$

Let $S^{[\mathfrak{p}]}(t) = \sum_{n \geq 0} S_n^{[\mathfrak{p}]} t^n$ be the gf enumerating the set of binary words $\left\{ w \in \mathcal{L}^{[\mathfrak{p}]} : |w|_0 \leq |w|_1 \right\}$ according to the number of bits 1

- if $n_1^{[\mathfrak{p}]} = n_0^{[\mathfrak{p}]} + 1$:

$$S^{[\mathfrak{p}]}(t) = \frac{2C^{[\mathfrak{p}]}(t)}{\sqrt{Q(t)} \left(\sqrt{C^{[\mathfrak{p}]}(t)} + \sqrt{Q(t)} \right)}$$

where $Q(t) = (1 - 4t) C^{[\mathfrak{p}]}(t)^2 + 4t^{n_1^{[\mathfrak{p}]}}$

Theorem 2: the case $n_0^{[\mathfrak{p}]} - n_1^{[\mathfrak{p}]} = 1$

- if $n_0^{[\mathfrak{p}]} = n_1^{[\mathfrak{p}]} + 1$:

$$S^{[\mathfrak{p}]}(t) = \frac{2C^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_1^{[\mathfrak{p}]}})}{\sqrt{Q(t)}\left(C^{[\mathfrak{p}]}(t) - 2t^{n_1^{[\mathfrak{p}]}} + \sqrt{Q(t)}\right)}$$

where $Q(t) = (1-4t)C^{[\mathfrak{p}]}(t)^2 + 4t^{n_0^{[\mathfrak{p}]}}C^{[\mathfrak{p}]}(t)$

Theorem 2: the case $n_0^{[\mathfrak{p}]} - n_1^{[\mathfrak{p}]} = 0$

- if $n_1^{[\mathfrak{p}]} = n_0^{[\mathfrak{p}]}$:

$$S^{[\mathfrak{p}]}(t) = \frac{2C^{[\mathfrak{p}]}(t)^2}{\sqrt{Q(t)}\left(C^{[\mathfrak{p}]}(t) - t^{n_0^{[\mathfrak{p}]}} + \sqrt{Q(t)}\right)}$$

where $Q(t) = (1-4t)C^{[\mathfrak{p}]}(t)^2 + 2t^{n_0^{[\mathfrak{p}]}}C^{[\mathfrak{p}]}(t) + t^{2n_0^{[\mathfrak{p}]}}$

Proof.

Observe that $S^{[\mathfrak{p}]}(t) = R^{[\mathfrak{p}]}(t, 1)$, or, equivalently, that $S_n^{[\mathfrak{p}]} = \sum_{k=0}^{n} R_{n,k}^{[\mathfrak{p}]}$ and apply the fundamental rule with $f_k = 1$. $\square$

Theorem 3: the case $n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]} = 1$

Let $L^{[\mathfrak{p}]}(t) = \sum_{n\geq 0} L_n^{[\mathfrak{p}]} t^n$ be the gf enumerating the set of binary words $\left\{w \in \mathcal{L}^{[\mathfrak{p}]} : |w|_0 \leq |w|_1\right\}$ according to the length

- if $n_1^{[\mathfrak{p}]} = n_0^{[\mathfrak{p}]} + 1$:

$$L^{[\mathfrak{p}]}(t) = \frac{2tC^{[\mathfrak{p}]}(t^2)^2}{\sqrt{Q(t)}\left((2t-1)C(t^2) + \sqrt{Q(t)}\right)}$$

where $Q(t) = C^{[\mathfrak{p}]}(t^2)\left((1-4t^2)C^{[\mathfrak{p}]}(t^2) + 4t^{2n_1^{[\mathfrak{p}]}}\right)$

Theorem 3: the case $n_0^{[\mathfrak{p}]} - n_1^{[\mathfrak{p}]} = 1$

- if $n_0^{[\mathfrak{p}]} = n_1^{[\mathfrak{p}]} + 1$:

$$L^{[\mathfrak{p}]}(t) = \frac{2t\sqrt{C^{[\mathfrak{p}]}(t^2)}(t^{2n_1^{[\mathfrak{p}]}} - C^{[\mathfrak{p}]}(t^2))}{\sqrt{Q(t)}\left((1-2t)C^{[\mathfrak{p}]}(t^2) + B(t) - \sqrt{C^{[\mathfrak{p}]}(t^2)Q(t)}\right)}$$

where $Q(t) = (1-4t^2)C^{[\mathfrak{p}]}(t^2) + 4t^{2n_0^{[\mathfrak{p}]}}$ and $B(t) = 2t^{n_0^{[\mathfrak{p}]}+n_1^{[\mathfrak{p}]}}$

Theorem 3: the case $n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]} = 0$

- if $n_1^{[\mathfrak{p}]} = n_0^{[\mathfrak{p}]}$:

$$L^{[\mathfrak{p}]}(t) = \frac{2tC^{[\mathfrak{p}]}(t^2)^2}{\sqrt{Q(t)}\left((2t-1)C(t^2) - t^{2n_0^{[\mathfrak{p}]}} + \sqrt{Q(t)}\right)}$$

where $Q(t) = (1-4t^2)C^{[\mathfrak{p}]}(t^2)^2 + 2t^{2n_0^{[\mathfrak{p}]}}C^{[\mathfrak{p}]}(t^2) + t^{4n_0^{[\mathfrak{p}]}}$

Theorem 3: proof

Proof.

Observe that the application of generating function $R^{[\mathfrak{p}]}(t,w)$ as

$$R^{[\mathfrak{p}]}\left(tw,\frac{1}{w}\right)=\sum_{n,k\in\mathbb{N}}R^{[\mathfrak{p}]}_{n,k}t^{n}w^{n-k}$$

entails that $[t^r w^s]R^{[\mathfrak{p}]}\left(tw,\frac{1}{w}\right)=R^{[\mathfrak{p}]}_{r,r-s}$ which is the number of binary words with r bits 1 and s bits 0. To enumerate according to the length let $t=w$, therefore

$$L^{[\mathfrak{p}]}(t)=\sum_{n\geq 0}L^{[\mathfrak{p}]}_{n}t^{n}=R^{[\mathfrak{p}]}\left(t^{2},\frac{1}{t}\right)$$

$\square$

Outline

Formulae for classes of patterns

- for $\mathfrak{p} = 1^{j+1}0^j$ we have:

$$S^{[\mathfrak{p}]}(t) = \frac{2}{\sqrt{Q(t)}\left(1 + \sqrt{Q(t)}\right)}, \quad Q(t) = 1 - 4t + 4t^{j+1}$$

- for $\mathfrak{p} = 0^{j+1}1^j$ we have:

$$S^{[\mathfrak{p}]}(t) = \frac{2(1 - t^j)}{\sqrt{Q(t)}\left(1 - 2t^j + \sqrt{Q(t)}\right)}, \quad Q(t) = 1 - 4t + 4t^{j+1}$$

- for $\mathfrak{p} = 1^j0^j$ and $\mathfrak{p} = 0^j1^j$ we have:

$$S^{[\mathfrak{p}]}(t) = \frac{2}{\sqrt{Q(t)}\left(1 - t^j + \sqrt{Q(t)}\right)}, \quad Q(t) = 1 - 4t + 2t^j + t^{2j}$$

Formulae for classes of patterns

- for $\mathfrak{p} = (10)^j 1$ we have:

$$S^{[\mathfrak{p}]}(t) = \frac{2(1 - t^{j+1})}{1 - 4t + 3t^{j+1} + \sqrt{Q(t)}}$$

where $Q(t) = 1 - 4t + 2t^{j+1} + 4t^{j+2} - 3t^{2j+2}$

- for $\mathfrak{p} = (01)^j 0$ we have:

$$S^{[\mathfrak{p}]}(t) = \frac{2(1 - t^j - t^{j+1} + t^{2j+1})}{\sqrt{Q(t)}\left(1 - 2t^j + t^{j+1} + \sqrt{Q(t)}\right)}$$

where $Q(t) = 1 - 4t + 2t^{j+1} + 4t^{j+2} - 3t^{2j+2}$

Series development for $S^{[1^{j+1}0^j]}(t)$

j/n	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	1	3	7	15	31	63	127	255	511	1023	2047	4095
2	1	3	10	32	106	357	1222	4230	14770	51918	183472	651191
3	1	3	10	35	123	442	1611	5931	22010	82187	308427	1162218
4	1	3	10	35	126	459	1696	6330	23806	90068	342430	1307138
5	1	3	10	35	126	462	1713	6415	24205	91874	350406	1341782
6	1	3	10	35	126	462	1716	6432	24290	92273	352212	1349768
7	1	3	10	35	126	462	1716	6435	24307	92358	352611	1351574
8	1	3	10	35	126	462	1716	6435	24310	92375	352696	1351973

$$\begin{aligned}[t^3]S^{[110]}(t) = \big|\{&111, 0111, 1011, 00111, 01011, 10011, 10101, 000111,\\ &001011, 010011, 010101, 100011, 100101, 101001, 101010\}\big| = 15\end{aligned}$$

Table: Some series developments for $S^{[1^{j+1}0^j]}(t)$ and the set of words with $n = 3$ bits 1, avoiding pattern $\mathfrak{p} = 110$, so $j = 1$ in the family; moreover, for $j = 1$ the sequence corresponds to $A000225$, for $j = 2$ the sequence corresponds to $A261058$.

formulae for classes of patterns

- for $\mathfrak{p} = 1^{j+1}0^j$ we have:

$$L^{[\mathfrak{p}]}(t) = \frac{2t}{\sqrt{Q(t)}\left(2t - 1 + \sqrt{Q(t)}\right)}, \quad Q(t) = 1 - 4t^2 + 4t^{2(j+1)}$$

- for $\mathfrak{p} = 0^{j+1}1^j$ we have:

$$L^{[\mathfrak{p}]}(t) = \frac{2t(t^{2j} - 1)}{\sqrt{Q(t)}\left(1 - 2t + 2t^{2j+1} - \sqrt{Q(t)}\right)}, \quad Q(t) = 1 - 4t^2 + 4t^{2(j+1)}$$

- for $\mathfrak{p} = 1^j0^j$ and $\mathfrak{p} = 0^j1^j$ we have:

$$L^{[\mathfrak{p}]}(t) = \frac{2t}{\sqrt{Q(t)}\left(-1 + 2t - t^{2j} + \sqrt{Q(t)}\right)}, \quad Q(t) = 1 - 4t^2 + 2t^{2j} + t^{4j}$$

formulae for classes of patterns

- for $\mathfrak{p} = (10)^j 1$ we have:

$$L^{[\mathfrak{p}]}(t) = \frac{2t(t^{2j+2} - 1)}{1 - 4t^2 + 3t^{2j+2} + (2t-1)\sqrt{Q(t)}}$$

where $Q(t) = 1 - 4t^2 + 2t^{2j+2} + 4t^{2j+4} - 3t^{4j+4}$

- for $\mathfrak{p} = (01)^j 0$ we have:

$$L^{[\mathfrak{p}]}(t) = \frac{2t(t^{2j+2} - 1)(t^{2j} - 1)}{\sqrt{Q(t)}\left(t^{2j+2} - 2t^{2j+1} + 2t - 1 + \sqrt{Q(t)}\right)}$$

where $Q(t) = 1 - 4t^2 + 2t^{2j+2} + 4t^{2j+4} - 3t^{4j+4}$

Series development for $L^{[1^{j+1}0^j]}(t)$

j/n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	3	3	7	7	15	15	31	31	63	63	127	127	255
2	1	1	3	4	11	15	38	55	135	201	483	736	1742	2699	6313
3	1	1	3	4	11	16	42	63	159	247	610	969	2354	3802	9117
4	1	1	3	4	11	16	42	64	163	255	634	1015	2482	4041	9752
5	1	1	3	4	11	16	42	64	163	256	638	1023	2506	4087	9880
6	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4095	9904
7	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4096	9908

Table: Some series developments for $L^{[1^{j+1}0^j]}(t)$; moreover, for $j = 1$ the sequence corresponds to $A052551$.

Closed formulae for particular cases

When the parameter j for a pattern $\mathfrak{p}$ assumes values 0 and 1 it is possible to find closed formulae for coefficients $S_n^{[\mathfrak{p}]}$ and $L_n^{[\mathfrak{p}]}$; moreover, in a recent submitted paper we give combinatorial interpretations, in terms of inversions in words and boxes occupancy, too.

$S_n^{[\mathfrak{p}]}$

$j/\mathfrak{p}$	$1^{j+1}0^j$	$0^{j+1}1^j$	1^j0^j
0	$[\![n=0]\!]$	1	$\binom{2n+1}{n}$
1	$2^{n+1}-1$	$(n+2)2^{n-1}$	$n+1$

Closed formulae for particular cases

$L_{2m}^{[\mathfrak{p}]}$

$j/\mathfrak{p}$	$1^{j+1}0^j$	$0^{j+1}1^j$	1^j0^j
0	$[\![n=0]\!]$	1	$2^{2m-1}+\frac{1}{2}\binom{2m}{m}$
1	$2^{m+1}-1$	$F_{2m+3}-2^m$	$m+1$

$L_{2m+1}^{[\mathfrak{p}]}$

$j/\mathfrak{p}$	$1^{j+1}0^j$	$0^{j+1}1^j$	1^j0^j
0	0	1	2^{2m-1}
1	$2^{m+1}-1$	$F_{2m+3}-2^{m+1}$	$m+1$

Summary

Key points

- split $F(t)$ in $F^{[\mathfrak{p}]}(x,y)$ to account for bits 1 and 0
- $R^{[\mathfrak{p}]}$ and $R^{[\bar{\mathfrak{p}}]}$ are both RA $\leftrightarrow$ $\mathfrak{p}$ is a Riordan pattern.
- requiring $|w|_0 \leq |w|_1$ entails

$$S^{[\mathfrak{p}]}(t) = R^{[\mathfrak{p}]}(t,1) \to [t^n]S^{[\mathfrak{p}]}(t) = \left| \left\{ w \in \mathcal{L}^{[\mathfrak{p}]} : \begin{array}{l} |w|_1 = n \\ |w|_0 \leq |w|_1 \end{array} \right\} \right|$$

$$L^{[\mathfrak{p}]}(t) = R^{[\mathfrak{p}]}\left(t^2, \frac{1}{t}\right) \to [t^n]L^{[\mathfrak{p}]}(t) = \left| \left\{ w \in \mathcal{L}^{[\mathfrak{p}]} : \begin{array}{l} |w| = n \\ |w|_0 \leq |w|_1 \end{array} \right\} \right|$$

Outlook

- provide combinatorial interpretations for both pattern classes $(10)^j1$ and $(01)^j0$, at least for $j \in \{0, 1\}$
- conjecture: when $j > 1$ in pattern classes it seems that $R^{[\mathfrak{p}]}$ is a binomial transformation
- build the Riordan graph for both RAs $R^{[\mathfrak{p}]}$ and $R^{[\bar{\mathfrak{p}}]}$ to study the meaning of pattern avoidance at graph level

고맙습니다