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Engineering Structures 26 (2004) 335–346

ENGINEERING STRUCTURES

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Evaluation of equivalent damping ratio of a structure with added dampers

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Received 21 October 2002; received in revised form 15 April 2003; accepted 17 September 2003

Abstract

The purpose of this study is to propose a new method for evaluating the equivalent damping ratio of a structure with supplemental damping devices to assess the vibration control effect quantitatively. The modal-energy-formed Lyapunov function is defined first, and the Riccati matrix and damping rate parameter are derived. Then from its analogy with the definition of a viscous modal damping ratio, the equivalent damping ratio is defined. The proposed approach is applied to estimate the equivalent damping ratio of a structure with various added damping devices. Then a closed-form solution is derived to obtain an equivalent modal damping ratio without carrying out numerical analysis. Results from the numerical analysis verify that the proposed method provides an equivalent damping ratio of a structure equipped with nonlinear as well as linear damping devices quite accurately.

Keywords: Equivalent damping; Supplemental damping devices; Lyapunov function; Seismic analysis

1. Introduction

The damping in a structure, although contributing significantly to dynamic responses, is difficult to quantify in mathematical form unlike other structural properties such as mass and stiffness. Various methods have been developed for estimating equivalent damping ratio from vibration test results [1,2]. In addition, eigenvalue analysis of a system matrix obtained from the system identification technique [3] and ARMA model can provide proper damping ratios [4].

Recently various types of damping devices, both passive and active, have been applied in structures to reduce excessive vibration [5]. For the purpose of investigating the control effects of damping devices, much research adopts the concept of equivalent damping ratio. Particularly, if the amount of damping ratio required to reduce the response of a structure to a desirable level is given, and the control efficiency of a damping device can be interpreted with a damping ratio, the design procedure of a damping device can be much simplified. Hartog [6] calculated the increase in damping ratio of a primary structure with a tuned mass damper as a function of mass ratio. Chang et al. [7] applied the modal-strain-energy method to assess the equivalent damping ratio of structures with viscoelastic dampers. Li and Reinhorn [8] derived the damping ratio of a structure with supplemental friction dampers through identification procedure using acceleration response transfer functions. In the NEHRP Guidelines for the Seismic Rehabilitation of Buildings (FEMA 273) [9], the ratio of dissipated energy to conserved energy is recommended to estimate the damping ratio of a structure with passive energy dissipation devices.

For active control, the procedure for evaluating damping ratios depends on control algorithms. If control forces are generated by linear control laws, damping ratios can be easily obtained from eigenvalue analysis. In the case of nonlinear control laws,

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0141-0296/$ - see front matter © 2003 Published by Elsevier Ltd.
doi:10.1016/j.engstruct.2003.09.014

however, eigenvalue analysis cannot be applicable and procedures such as the half power band-width method, the system identification technique, and the ratio of dissipated to conserved energy need to be applied to estimate the equivalent damping ratio. These methods, however, have their restrictions with application; for example, half power method can be applied when damping is relatively small, and the method based on dissipated to conserved energy can be applied accurately only when excitation is harmonic force.

The purpose of this study is to propose a new and generalized approach to evaluate the equivalent damping ratio of a structure with any type of supplemental damping device. The equivalent damping ratio is evaluated using the Lyapunov function and its derivative. In the process, it is assumed that:

1. the response of a structure is stationary random process;
2. the damping devices do not affect the mode shapes of the structure; and
3. the structure has proportional inherent damping.

These assumptions can be justified by the fact that supplemental damping devices are minor elements in building structures and proportional damping can describe the mechanism of energy dissipation. To show the effectiveness of the proposed approach, the equivalent modal damping ratios of a structure with linear viscous dampers (VDs), active mass driver (AMD), and friction dampers (FDs) are derived, and the results are compared with those obtained by conventional eigenvalue analysis. Also closed-form formulas are derived to quantify the equivalent damping ratio of a structure with added dampers without carrying out time-history analysis. It should be noted that the proposed approach has little advantage in predicting equivalent damping of a linear system compared to conventional method. However, in structures with nonlinear damping devices, the proposed procedure can provide a convenient tool for estimation of the damping effect.

2. Derivation of equivalent damping ratios

2.1. Equation of motion

The equation of motion of a multi-degrees-of-freedom (MDOF) system subjected to dynamic environmental load $f(t)$ and control force $u(t)$ is given by

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = L_c u(t) + L_e f(t), \tag{1}$$

where M, C and K denote the mass, damping and stiffness matrices, respectively; $x(t)$ is the displacement vector; and L_c and L_e represent the location vector of control force contributed from supplemental damping devices and the influence vector of excitation, respectively. Using mode-superposition method, Eq. (1) can be transformed to the following modal equation:

$$\ddot{\eta}_i(t) + 2\xi_i\omega_i\dot{\eta}_i(t) + \omega_i^2\eta(t) = \phi_i^T L_c u(t) + \phi_i^T L_e f(t),$$
$$x(t) = \sum_i^n \phi_i \eta_i(t) \tag{2}$$

where η_i, ξ_i, ω_i, and ϕ_i denote the displacement, damping ratio, natural frequency, and eigenvector of the ith mode, respectively. The eigenvector is mass normalized. The state-space expression of Eq. (2) becomes

$$\dot{z}_i(t) = A_i z_i(t) + B_i u(t) + H_i f(t),$$
$$z_i(t) = [\eta_i(t) \quad \dot{\eta}_i(t)]^T \tag{3}$$

where the matrices A_i, B_i, and H_i are given by

$$A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\omega_i\xi_i \end{bmatrix} \qquad B_i = \begin{bmatrix} 0 \\ \phi_i^T L_c \end{bmatrix}$$
$$H_i = \begin{bmatrix} 0 \\ \phi_i^T L_e \end{bmatrix}. \tag{4}$$

The Lyapunov function e_i is defined as follows:

$$e_i(t) = z_i^T(t) P_i z_i(t), \tag{5}$$

where P_i is arbitrarily chosen positive definite Riccati matrix. By differentiating both sides of Eq. (5) and substituting Eq. (3), we get

$$\dot{e}_i(t) = z_i^T(t)\left(A_i^T P_i + P_i A_i\right) z_i(t) + 2z_i^T(t) P_i B_i u(t) + 2z_i^T(t) P_i H_i f(t). \tag{6}$$

If positive definite Riccati matrix P_i and positive scalar α_i exist to satisfy the following Eq. (7), the derivative of the Lyapunov function can be expressed in autoregressive form and modal states can satisfy the stability condition, as shown in Eq. (8)

$$A_i^T P_i + P_i A_i = -\alpha_i P_i \tag{7}$$

$$\dot{e}_i(t) = -\alpha_i e_i(t) + 2z_i^T(t) P_i B_i u(t) + 2z_i^T(t) P_i H_i f(t). \tag{8}$$

Eq. (8) indicates that the amplitude of $e_i(t)$ decays exponentially with time without control force $u(t)$ and excitation $f(t)$. The property α_i is related to the damping rate at which free vibration decays.

2.2. Calculation of P and α

Solving the eigenvalue problem of A_i leads to the following relationship:

$$A_i = \Phi\Lambda\Phi^{-1} \tag{9}$$

in which $\Phi = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. The eigenvalues of A_i, λ_1 and λ_2 are

$$\lambda_{1,2} = -\xi_i\omega_i \pm \omega_i\sqrt{1-\xi_i^2}\,i. \tag{10}$$

Substitution of Eq. (9) for A_i in Eq. (7) gives

$$\left(\Phi^{-1}\right)^T \Lambda \Phi^T P_i + P_i \Phi \Lambda \Phi^{-1} = -\alpha_i P_i. \quad (11)$$

If both sides of Eq. (11) are pre-multiplied by Φ^T and post-multiplied by Φ, Eq. (11) can be written in compact form as

$$\Lambda \bar{P} + \bar{P}\Lambda = -\alpha_i \bar{P} \quad (12)$$

in which $\bar{P} = \Phi^T P_i \Phi$. Eq. (12) can be separated into element-to-element equations; i.e. the elements $\bar{p}_{ij(i,j=1,2)}$ of $\bar{P}$ are determined by solving the following equations:

$$2\lambda_1 \bar{p}_{11} = -\alpha_i \bar{p}_{11} \quad (13)$$

$$2\lambda_2 \bar{p}_{22} = -\alpha_i \bar{p}_{22} \quad (14)$$

$$(\lambda_1 + \lambda_2)\bar{p}_{12} = -\alpha_i \bar{p}_{12}. \quad (15)$$

If all the elements of $\bar{P}$ are nonzero, there is no constant α_i to satisfy Eq. (13)–(15) simultaneously. Positive definite matrix P_i and the constant α_i can be obtained as follows by setting $\bar{p}_{11} = \bar{p}_{22} = 0$ and $\bar{p}_{12} \neq 0$:

$$\alpha_i = -(\lambda_1 + \lambda_2) = 2\xi_i \omega_i \quad (16)$$

$$P_i = \left(\Phi^T\right)^{-1} \bar{P} \Phi^{-1} = \frac{\bar{p}_{12}}{8\omega_i^2(1-\xi_i^2)} \begin{bmatrix} \omega_i^2 & \xi_i \omega_i \\ \xi_i \omega_i & 1 \end{bmatrix}. \quad (17)$$

Because the variation of $\bar{p}_{12}$ has no effect on the physical meaning of e_i ($\bar{p}_{12}$ merely affects the relative magnitude of e_i), $\bar{p}_{12}$ can be set to $8\omega_i^2(1-\xi_i^2)$ without loss of generality, then P_i and e_i become

$$P_i = \begin{bmatrix} \omega_i^2 & \xi_i \omega_1 \\ \xi_i \omega_1 & 1 \end{bmatrix} \quad (18)$$

$$e_i(t) = \omega_i^2 \eta_i^2(t) + 2\xi_i \omega_i \eta_i(t) \dot{\eta}_i(t) + \dot{\eta}_i^2(t). \quad (19)$$

Since the damping ratio ξ_i of a typical civil structure is much smaller than 1, the second term in the left-hand side of Eq. (19) is very small, and therefore $e_i(t)$ is positive. In this study $e_i(t)$ is the modified energy of the ith mode. The term modified energy is used because $e_i(t)$ includes the correlation between modal displacement and velocity.

Fig. 1 compares the total energy of a single-degree-of-freedom (SDOF) system undergoing free vibration, consisting of the strain energy, and kinetic energy, with the previously defined modified energy. The mass, stiffness, and damping of the system are given as 15 kg, 770 N/m, and 10%, respectively. It can be observed that the modified energy decays exponentially with time as expected, but the total energy decays with fluctuation. Therefore, the effect of damping is more apparent in the dissipation of modified energy than in the dissipation of the total energy.

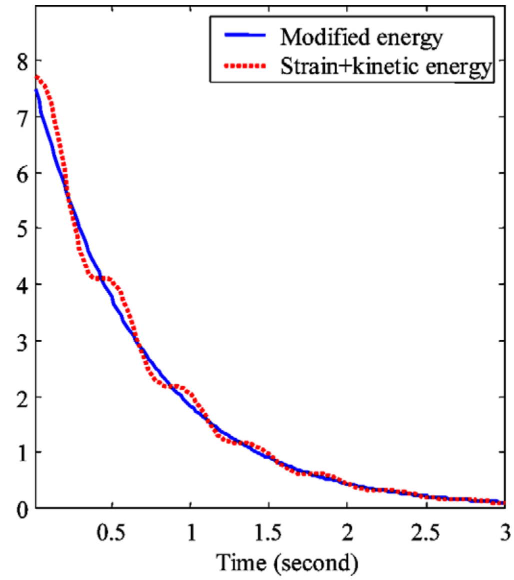

Fig. 1. Modified energy and strain+kinetic energy.

2.3. Equivalent damping ratios

Eq. (8) indicates that the control force $u(t)$ generated by supplemental damping devices can play the same role as α_i which decreases the Lyapunov function e_i exponentially with time. For the normalization of the effect of $u(t)$ in terms of α_i, the second term in the right-hand side of Eq. (8) can be expressed as follows with the introduction of another parameter, α_{ia}:

$$u_i(t) = -\alpha_{ia} e_i(t) \quad (20)$$

in which $u_i(t) = 2z_i^T(t) P_i B_i u(t)$ and the parameter α_{ia} is a function of time as are the Lyapunov function $e_i(t)$ and $u_i(t)$. The damping ratio introduced by damping devices can be obtained as follows with reference to Eq. (16):

$$\xi_{ia} = \frac{\alpha_{ia}}{2\omega_i}. \quad (21)$$

The proposed equation for the damping ratio is a function of time while the damping ratio in typical structural dynamic problem has constant value. This inconsistency is natural considering the dynamic characteristics of damping devices, whose damping effects vary with time. Therefore a constant representative value for time-dependent damping ratio is required for efficient evaluation of damping capacity and for simplification of analysis. This can be realized by taking the mean value of both sides of Eq. (20):

$$E[u_i(t)] = -\alpha_{i_{eq}} E[e_i(t)] \quad (22)$$

where $E[\cdot]$ is the function of expectation which averages the control effect of damping devices. The constant value of $\alpha_{i_{eq}}$ computed from Eq. (22) is used to obtain the constant damping ratio as follows:

$$\xi_{i_{eq}} = \frac{\alpha_{i_{eq}}}{2\omega_i}. \tag{23}$$

Finally the equation of motion, Eq. (2), can be rewritten as follows using the equivalent damping ratio ξ_{ieq} computed from Eq. (23):

$$\ddot{\eta}_i(t) + 2(\xi_i + \xi_{i_{eq}})\omega_i\dot{\eta}_i(t) + \omega_i^2\eta(t) = \phi_i^T L_e f(t). \tag{24}$$

3. Closed-form solutions for the equivalent damping ratio

The proposed method is powerful and versatile in the sense that it can provide equivalent damping ratios associated with higher modes, and that it can be applied to damping devices controlled by the nonlinear as well as the linear control law. The method, however, has a shortcoming in that it requires dynamic time-history analysis to evaluate the equivalent modal damping ratio. This will be troublesome and time consuming especially in the preliminary design stage.

In this section, closed-form formulas are derived to quantify the equivalent damping ratio of a structure with added dampers without carrying out time-history analysis. The derivation is based on the well-known knowledge that the dynamic response of typical building structures is represented well by the fundamental mode. A MDOF system is transformed to an equivalent SDOF system. Then the effect of damping device on this fundamental mode is estimated by using the proposed method.

3.1. Equivalent SDOF model for a MDOF system

To transform a MDOF system into an equivalent SDOF system, the displacement vector $x(t)$ of a MDOF system is expressed approximately as

$$x(t) = \phi d(t), \tag{25}$$

where ϕ is a vector of deflection shape for the fundamental mode normalized to the top-story displacement $d(t)$. Then the equation of motion of an equivalent SDOF system is obtained as

$$M^*\ddot{d}(t) + C^*\dot{d}(t) + K^*d(t) = \phi^T L_u u(t) + \phi^T L_e f(t), \tag{26}$$

where $M^* = \phi^T M\phi$, $C^* = \phi^T C\phi$, and $K^* = \phi^T K\phi$. Dividing both sides of the Eq. (26) by M^*, we obtain

$$\ddot{d}(t) + 2\xi_o\omega_o\dot{d}(t) + \omega_o^2 d(t) = \frac{\phi^T L_u}{M^*}u(t) + \frac{\phi^T L_e}{M^*}f(t), \tag{27}$$

where $\omega_o^2 = K^*/M^*$ and $\xi_o = C^*/2\omega_o M^*$. The damping ratio increased by a damping device in the equivalent SDOF system, ξ_{eq}, can be derived as follows using Eq. (23):

$$\xi_{\text{eq}} = -\frac{E\left[z(t)^T PBu(t)\right]}{\omega_o E\left[z(t)^T Pz(t)\right]}, \tag{28}$$

where $z(t)^T = \begin{bmatrix} d(t) & \dot{d}(t) \end{bmatrix}^T$, $P = \begin{bmatrix} \omega_o^2 & \xi_o\omega_o \\ \xi_o\omega_o & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 0 \\ \frac{\phi^T L_u}{M^*} \end{bmatrix}$.

To evaluate the damping ratio of the equivalent SDOF system, determining the shape vector is essential. In this study ϕ is assumed to be the shape vector corresponding to the deflected shape under the action of a statically applied lateral load with an inverted triangular distribution pattern represented as follows [1]:

$$F_j = (V_b - F_t)\frac{w_j h_j}{\sum_{i=1}^N w_i h_i}, \tag{29}$$

where F_j is the seismic story force at the jth story, h_j and w_j are the height and weight of the jth story, respectively, and $V_b$is the base shear. The force at the top floor is increased by an additional force F_t, which has a different value according to the fundamental natural period. The National Building Code of Canada (NBCC) [10] recommends:

$$F_t = \begin{cases} 0 & T_1 \le 0.7 \\ 0.07T_1V_b & 0.7 < T_1 \le 3.6 \\ 0.25V_b & T_1 \ge 3.6 \end{cases}. \tag{30}$$

3.2. Derivation of closed-form solutions

Structural responses for dynamic loads such as earthquakes or wind loads have statistical characteristics; i.e. responses can be described by the mean and standard deviation. Considering that forces generated from damping devices are dependent on the structural responses, the equivalent damping contributed from these damping forces also has statistical characteristics. In this sense the method proposed in this study begins with taking the expected responses to obtain mean values based on the assumption that responses are stationary and Gaussian. Then statistical properties of responses are obtained and closed-form formulas for equivalent damping ratios are derived.

It is assumed that the top-floor displacement d and velocity $\dot{d}$ have the following statistical characteristics:

$$E\left[d^2(t)\right] = \sigma_d^2 \tag{31}$$

$$E\left[\dot{d}^2(t)\right] = \sigma_{\dot{d}}^2 = \omega_o^2\sigma_d^2 \tag{32}$$

$$E\left[d(t)\cdot\dot{d}(t)\right] = 0, \tag{33}$$

where σ_d and $\sigma_{\dot{d}}$ are standard deviations of displacement and velocity, respectively. Using the above equations, the denominator of Eq. (28) becomes

$$\omega_o E\left[z(t)^T P z(t)\right] = \omega_o\left[\omega_o^2 d^2(t) + 2\xi_o\omega_o d(t)\dot{d}(t) + \dot{d}^2(t)\right] = 2\omega_o^3\sigma_d^2. \tag{34}$$

In order to evaluate the numerator in Eq. (28), the control force $u(t)$ should be known. As $u(t)$ is generally a function of the displacement and velocity responses, equivalent damping ratio is evaluated for the following cases for $u(t)$:

- Case 1: $u(t)$is linearly proportional to the relative velocity between the ends of a device.
- Case 2: $u(t)$is a linear function of states.
- Case 3: $u(t)$is a constant multiplied by the sign of the relative velocity between the ends of a device.

In Case 1 and 2 the damping device adds stiffness or viscosity to the structure. Case 3 corresponds to the FD or bang-bang controlled AMD. The control force for each case generated from the damping device installed at the jth inter-story or on the jth floor is given by

- For Case 1:

$$u_j(t) = c_{oj}\dot{d}(t)\delta_{rj}\cos\theta_j\ ; \tag{35}$$

- For Case 2:

$$u(t) = -Gz(t) = -G_1\phi d(t) - G_2\phi\dot{d}(t); \tag{36}$$

- For Case 3:

$$u_j(t) = u_{\max j}\mathrm{sgn}\left[\dot{d}(t)\delta_{rj}\right]\cos\theta_j; \tag{37}$$

in which θ_j is the angle between the axis of the jth device and the floor at which the device is installed. c_{oj} and $u_{\max j}$ are viscosity and maximum control force of device j, respectively. The relative displacement between the ends of device j along the device axis, δ_{rj} is determined based on the deflection shape vector ϕ, and sgn[·] is the sign function. G_1 and G_2 are gains for displacement feedback and velocity feedback, respectively. Substituting the expressions of P and B into the numerator of Eq. (28) leads to

$$E\left[z(t)^T PBu(t)\right] = \frac{1}{M^*}E\left[\left(\xi_o\omega_o d(t) + \dot{d}(t)\right)\phi^T L_u u(t)\right]. \tag{38}$$

The jth element of the vector $\phi^T L_u$ for Cases 1–3 is given by

$$\left[\phi^T L_u\right]_{jth\_element} = -\delta_{rj}\cos\theta_j. \tag{39}$$

Thus, the corresponding numerators become

- For Case 1:

$$\frac{1}{M^*}E\left[\left(\xi_o\omega_o d + \dot{d}\right)\cdot\sum_j(-\delta_{rj}\cos\theta_j)c_{oj}\dot{d}\delta_{rj}\cos\theta_j\right] = \frac{-1}{M^*}\sum_j E\left[c_{oj}\dot{d}^2\delta_{rj}^2\cos\theta_j^2\right] = \frac{-\omega_o^2\sigma_d^2}{M^*}\sum_j c_{oj}\delta_{rj}^2\cos\theta_j^2 \tag{40}$$

- For Case 2:

$$\frac{1}{M^*}E\left[\left(\xi_o\omega_o d(t)+\dot{d}(t)\right)\cdot\phi^T L_u\left(-G_1\phi d(t)-G_2\phi\dot{d}(t)\right)\right] = \frac{-1}{M^*}\sum_j E\left[\phi^T L_u\left(\xi_o\omega_o G_1\phi d^2(t)+G_2\phi\dot{d}^2(t)\right)\right] = \frac{-\phi^T L_u(\xi_o\omega_o G_1\phi+G_2\phi\omega_o^2)\sigma_d^2}{M^*} \tag{41}$$

- For Case 3:

$$\frac{1}{M^*}E\left[\left(\xi_o\omega_o d(t)+\dot{d}(t)\right)\cdot\sum_j(-\delta_{rj}\cos\theta_j)u_{\max j} \times \mathrm{sgn}\left[\dot{d}(t)\delta_{rj}\right]\cos\theta_j\right] = \frac{-1}{M^*}\sum_j E\left[u_{\max j}\left|\dot{d}(t)\right|\left|\delta_{rj}\right|\cos\theta_j^2\right] = \frac{-E\left[\left|\dot{d}\right|\right]}{M^*}\sum_j u_{\max j}\left|\delta_{rj}\right|\cos\theta_j^2. \tag{42}$$

The expectation of the absolute value of the variable x, which has Gaussian probability distribution with zero mean and standard deviationσ_x, is

$$E[|x|] = 2\int_0^\infty \frac{x}{\sqrt{2\pi}\sigma_x}\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\mathrm{d}x = \sqrt{\frac{2}{\pi}}\sigma_x. \tag{43}$$

Accordingly, $E[|\dot{x}|]$ becomes

$$E[|\dot{x}|] = \sqrt{\frac{2}{\pi}}\sigma_{\dot{x}} = \sqrt{\frac{2}{\pi}}\omega_o\sigma_x. \tag{44}$$

Now, using the denominators and numerators derived above, the following closed-form formulas for equivalent damping ratios of a structure with damping devices can be obtained for each case:

$$\xi_{eq} = \frac{1}{2M^*\omega_o}\sum_j c_{oj}\delta_{rj}^2\cos\theta_j^2 \text{ for Case 1;} \tag{45}$$

$$\xi_{eq} = \xi_o\frac{\phi^T L_u G_1\phi}{2M^*\omega_o^2} + \frac{\phi^T L_u G_2\phi}{2M^*\omega_o} \text{ for Case 2;} \tag{46}$$

$$\xi_{eq} = \frac{1}{\sqrt{2\pi}M^*\omega_o^2\sigma_d}\sum_j u_{\max j}\left|\delta_{rj}\right|\cos\theta_j^2 \text{ for Case 3.} \tag{47}$$

Eqs. (45) and (46) can be derived directly from the equations of the system dominated by the first mode. This means that the proposed method is suitable for the evaluation of the equivalent damping ratio of a linear system. While the equivalent damping ratio ξ_{eq} for Cases 1 and 2 can be easily determined only with information of ϕ and the properties or the gain of the damping devices, σ_d should be known for the evaluation of ξ_{eq} for Case 3. However, since σ_d is the standard deviation of top-floor displacement obtained by seismic analysis of the building-damper system, ξ_{eq} cannot be predicted in advance of seismic analysis for the control efficiency. Therefore, an iterative seismic analysis procedure is necessary for the evaluation of ξ_{eq} for Case 3. However, even in this case, the prediction of ξ_{eq} may be possible by using an approximate technique. Fu and Kasai [11] suggested the following simplified expression to represent the response variation due to the change in the damping ratio from $\xi_o$to ξ

$$\frac{S_d(\xi)}{S_d(\xi_o)} = \frac{\sqrt{1+25\xi_o}}{\sqrt{1+25\xi}}, \tag{48}$$

where S_d denotes the spectral displacement of a structure. Since the spectral response is approximately proportional to RMS response, with the help of Eq. (48), σ_d can be expressed as a function of σ_{d_o}:

$$\sigma_d = \sigma_{d_o}\frac{\sqrt{1+25\xi_o}}{\sqrt{1+25\left(\xi_o+\xi_{eq}\right)}}, \tag{49}$$

where d_o is the displacement of a structure without damping device. By using Eq. (49), Eq. (47) can be expressed as

$$\xi_{eq} = \frac{\sum_j u_{\max j}\left|\delta_{rj}\right|\cos\theta_j^2\sqrt{1+25\left(\xi_o+\xi_{eq}\right)}}{\sqrt{2\pi}M^*\omega_o^2\sigma_{d_o}\sqrt{1+25\xi_o}} = C_1\frac{\sqrt{1+25\left(\xi_o+\xi_{eq}\right)}}{\sqrt{1+25\xi_o}}, \tag{50}$$

in which

$$C_1 = \frac{\sum_j u_{\max j}\left|\delta_{rj}\right|\cos\theta_j^2}{\sqrt{2\pi}M^*\omega_o^2\sigma_{d_o}}.$$

From Eq. (50), ξ_{eq} can be obtained as

$$\xi_{eq} = \frac{25C_1^2 + C_1\sqrt{625C_1^2 + 4(1+25\xi_o)^2}}{2(1+25\xi_o)}. \tag{51}$$

4. Numerical analysis

4.1. Example 1—three-story shear building

Numerical analysis is performed with two-dimensional three-story shear buildings installed with damping devices such as VD, AMD and FD. It is assumed that every story unit is identically constructed. The mass and stiffness of each story unit are $m = 50$ ton and $k = 15000$ kN/m, respectively. Modal damping ratios of the building are assumed to be $\xi_{i(i=1,2,3)} = 0.02$. The mass-normalized mode shape vectors and natural frequencies of the model structures are listed in Table 1. White noise ground excitation is used as input earthquake load.

4.1.1. Linear VDs

A linear VD is installed in every inter-story of the model structure as shown in Fig. 2. Each VD has the same damping coefficient c_o. The modal damping ratios obtained by the proposed approach are compared with those computed by eigenvalue analysis in Fig. 3. The comparison of the analysis results indicates that the damping ratios obtained by the proposed approach are very close to those obtained by eigenvalue analysis. The slight difference between eigenvalue analysis and the proposed method results from the fact that, contrary to eigenvalue analysis which obtains the damping ratio by solving a mathematical equation, the proposed method derives the modal damping ratio through analyzing the structural responses induced by a specific excitation load.

4.1.2. AMD

Generally the control force generated by an AMD is dependent on the control law applied, and in this study the linear quadratic regulator (LQR) algorithm is used as a control law [12]. Fig. 4 depicts the model structure with an AMD located on the roof floor. The modal damping ratios are computed by the proposed

Table 1
Modal properties of the model structure

Mode		1	2	3
Damping ratio (%)		2.0	2.0	2.0
Frequency (Hz)		1.23	3.44	4.97
Mode shape	1st floor	0.0015	0.0033	0.0026
	2nd floor	0.0026	0.0015	−0.0033
	3rd floor	0.0033	−0.0026	0.0015

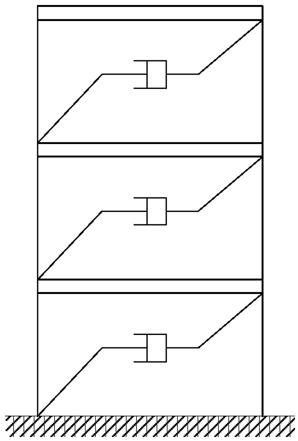

Fig. 2. Modified structure with VDs.

approach and by eigenvalue analysis, and the results are presented in Fig. 5. It can be observed that the damping ratios obtained by the proposed method are almost identical to those obtained using eigenvalue analysis. The results show that the proposed method can be applicable for a structure installed with active damping devices such as an AMD using LQR control law. It should be noted that an AMD using LQR, located on the roof floor, generally produces a nonproportional damping effect, which does not agree with the assumption that damping devices do not affect the mode shapes of the structure. Therefore, the error found in Fig. 5 seems to come from either the violation of the assumption or the same source of error observed in Fig. 3.

4.1.3. FDs

Fig. 6 describes the model structure with a FD at each inter-story. The FD dissipates vibration energy by Coulomb damping resulting from friction against sliding of two dry surfaces. The direction of the damping force opposes the sign of relative velocity $\dot{x}_j(t)$. The damping force produced by FD installed at the jth inter-story is represented as follows:

$$u_j(t) = -F_c \text{sgn}\left[\dot{x}_j(t)\right], \tag{52}$$

where F_c denotes the friction force of a damper.

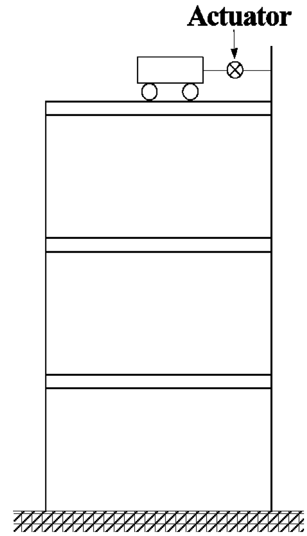

Fig. 4. Model structure with an AMD.

Fig. 7 presents the equivalent damping ratios of the model structure obtained by the proposed method for various friction forces of the dampers. FDs with the same property are installed in every inter-story of the structure. In Fig. 8, RMS and peak responses obtained by mode superposition method using the equivalent damping ratios obtained using Eq. (23) are compared to those computed by time history analysis of the integrated building-FD system. It can be observed from the results that the RMS responses are closer to those from time-history analysis than peak responses. This tendency is due to the fact that expectation is taken in Eq. (22) to evaluate equivalent damping ratios. However, absolute acceleration responses are much smaller than the exact value obtained by time-history analysis of the integrated model. This shows that FD, which always generates maximum friction force whose direction changes abruptly according to the relative velocity, is not so effective in the reduction of absolute acceleration response, and that the equivalent damping ratio cannot reflect this phenomenon. This tendency can be confirmed from unreasonably large equivalent damping ratios in higher modes which have more

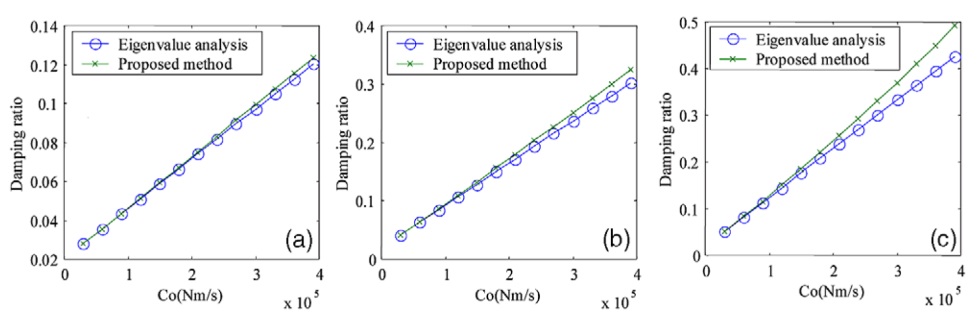

Fig. 3. Damping ratio of a structure with linear VD: (a) 1st mode; (b) 2nd mode; (c) 3rd mode.

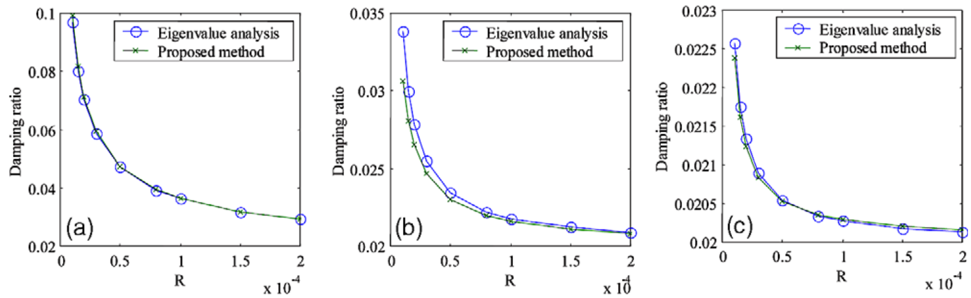

Fig. 5. Damping ratio of the model structure with AMD: (a) 1st mode; (b) 2nd mode; (c) 3rd mode.

effects on acceleration response than on displacement response. Therefore, the effects of FD with large maximum force on absolute acceleration responses may be overestimated if equivalent damping ratios are used.

4.2. Example 2—ten-story shear building

To verify the accuracy of the proposed equivalent SDOF model and the closed-form formulas for equivalent damping ratios, analyses are performed with a ten-story shear building. Damping devices such as VD, FD, and AMD using LQR algorithm are used. Linear VD and FD with the same properties are installed at every inter-story of the building, and an AMD is installed on the top floor. Each story unit of the building has identical lumped mass of 100 ton; story stiffness of $k_{i=1,\ldots,4} = 15000$ kN, $k_{i=5,6,7} = 10500$ kN, and $k_{i=8,9,10} = 7350$ kN. The inherent damping matrix is constructed so that all the modal damping ratios become 2%. White noise earthquake excitation is used as an input for the dynamic analysis.

Fig. 9 presents the maximum and RMS values for the top-floor displacement of the building equipped with VDs. The displacements are obtained for:

1. a ten-story shear building model integrated with damping devices (denoted as the *integrated model*);
2. an equivalent SDOF system using closed-form formula for the equivalent damping (denoted as the *closed-form formula*); and
3. a ten-story shear building model with equivalent damping ratios computed from Eq. (23) (denoted as the *MDOF system*).

The results show that, as expected, the maximum and RMS displacements decrease as the VD coefficient, c_o, increase. It also can be observed that the *MDOF system* reproduces almost the same results as those obtained from the *integrated model*, while the equivalent SDOF model using a closed-form formula shows a little discrepancy. The error in the equivalent SDOF model is large for small damping, and is caused by the reduction in system order.

Fig. 10 depicts the equivalent damping ratios evaluated by various methods such as:

1. eigenvalue analysis of the model structure with damping devices;
2. proposed method using Eq. (23);
3. equivalent SDOF system with the equivalent damping ratios obtained using Eq. (23); and finally
4. equivalent damping ratios computed by the closed-form formulas.

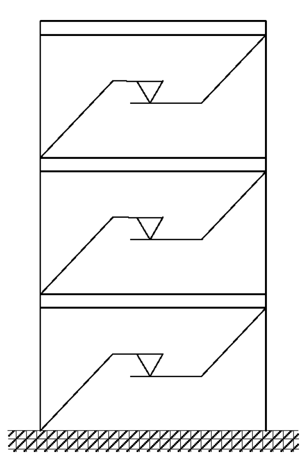

Fig. 6. Model structure with FDs.

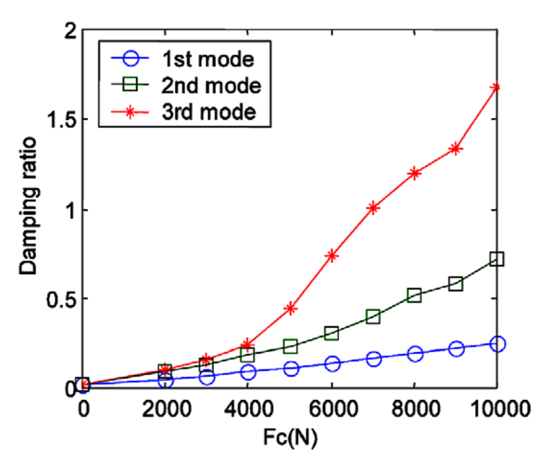

Fig. 7. Equivalent damping ratio of the model structure with FDs.

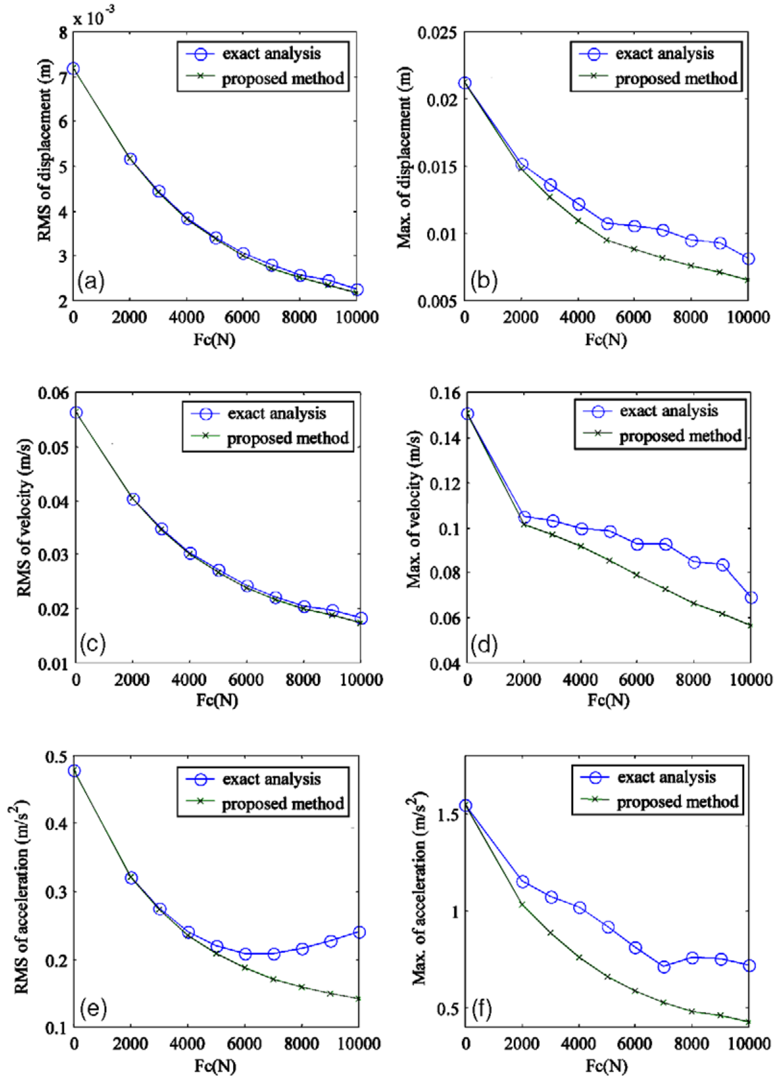

Fig. 8. Top-floor response of the model structure with FDs: (a) RMS displacement; (b) peak displacement; (c) RMS velocity; (d) peak velocity; (e) RMS acceleration; (f) peak acceleration.

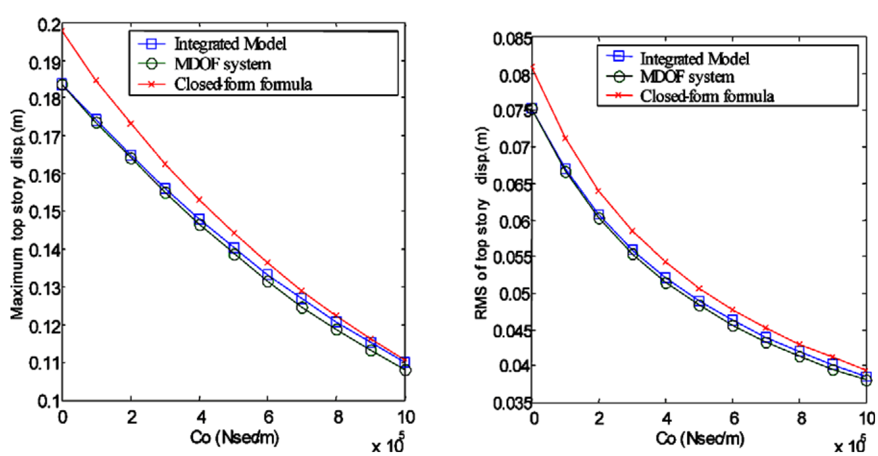

Fig. 9. Top-story displacement of the model structure with VDs: (a) maximum displacements; (b) RMS displacements.

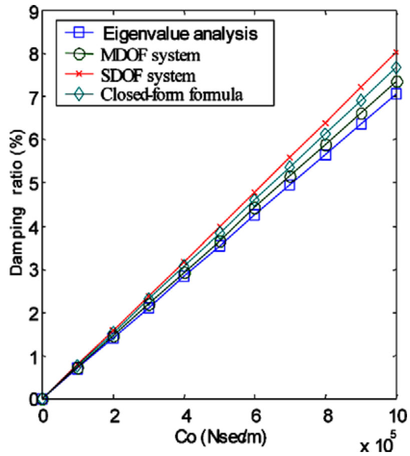

Fig. 10. Fundamental modal damping ratio of the model structure with various damping constants of added VDs.

As expected, the equivalent damping ratios increase with increasing c_o. The results show that the equivalent damping ratios for the fundamental mode, obtained using the closed-form formula (Eq. (24)), are close to the other equivalent damping ratios, which are obtained by eigenvalue analysis or earthquake analysis. It is interesting to note that the difference between responses becomes small with increasing c_o, while the difference between equivalent damping ratio obtained by the closed-form formula and by the *MDOF model* becomes large with increasing c_o. This fact together with the findings from Fig. 9 indicates that the variation of response due to the effect of damping decreases as the added damping increases.

The analysis results for the model structure with an AMD controlled by LQR at the top floor are shown in Fig. 11. The ratio of the top-story displacements of the controlled and uncontrolled structures is defined as the reduction ratio R_r. It can be observed from Fig. 11 that the reduction ratio R_r corresponding to the MDOF system with equivalent damping ratios and the equivalent SDOF model are very close to those computed from analysis of the actual MDOF system with AMD (the integrated model). The fundamental equivalent damping ratio of the model structure with various weighting factors for control force, presented in Fig. 12, shows that, as expected, the equivalent damping decreases as the weighting factor R increases. It also can be found that the proposed simplified methods can provide the fundamental equivalent damping ratio quite accurately.

Figs. 13 and 14 present the results for the structure with a FD in every story. Fig. 13 shows the maximum and the RMS responses of the model structure using the equivalent damping ratio proposed in this study. The RMS responses turn out to be closer to exact solutions than maximum responses are. It can be observed that when the value of F_c is small, the responses predicted by using the SDOF model overestimate the responses obtained from the integrated model. However, as the control force limits increase, this phenomenon is reversed, and the SDOF model underestimates the exact responses at the high control force. The responses obtained from the MDOF model form lower bound of the exact solution, and the discrepancy increases as the control force increases.

The phenomenon observed above can be explained by the change in equivalent damping presented in Fig. 14. In structures with nonlinear damping devices, the modal damping ratios cannot be obtained from eigenvalue analysis. Therefore the equivalent damping ratio estimated by the proposed method is assumed to be precise value. Fig. 14 shows that, compared with the fundamental damping ratio obtained in the MDOF system, equivalent damping ratios provided by the closed-form formula are underestimated, resulting in

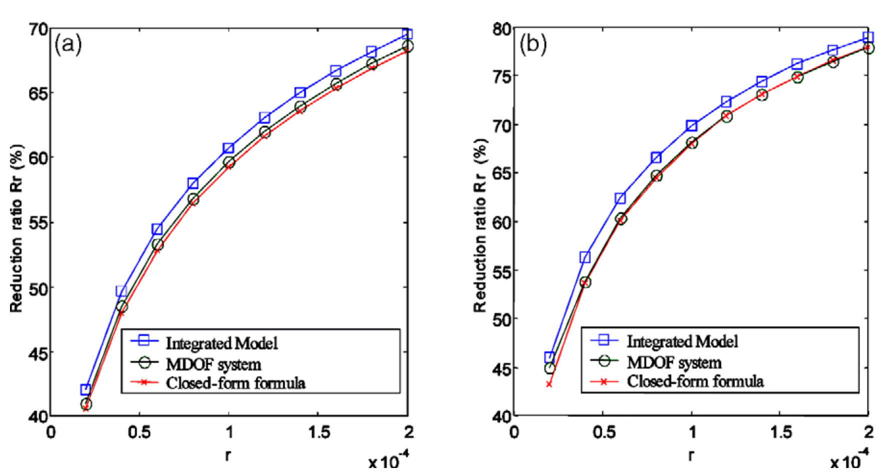

Fig. 11. Reduction ratio R_r for top-floor displacement of the model structure with an AMD using LQR: (a) maximum displacement; (b) RMS displacement.

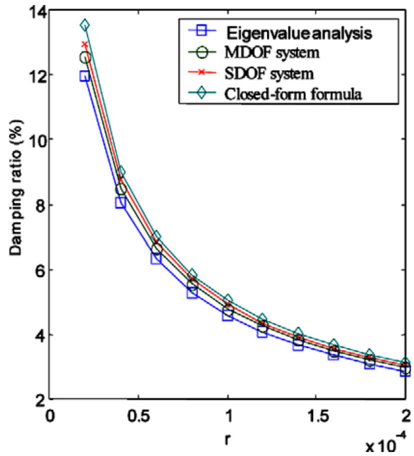

Fig. 12. Fundamental modal damping ratio of the model structure with an AMD using LQR.

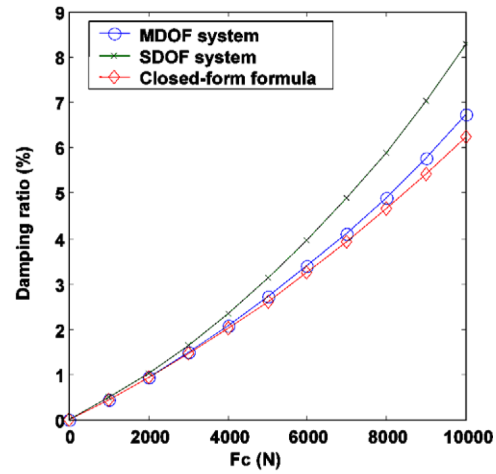

Fig. 14. Fundamental modal damping ratio of the model structure with FDs.

overestimation of responses. However, the equivalent damping ratios predicted in the equivalent SDOF structure using Eq. (23) rather deviate from those obtained in the MDOF system. Therefore, for a conservative design of a structure with Coulomb friction-type damping devices, the proposed closed-form formula is recommended to estimate the equivalent damping contributed from the added dampers at the preliminary analysis and design stages.

5. Conclusions

The objective of this study is to present a simple method for evaluating equivalent damping ratios of a structure equipped with supplemental damping devices to quantify their vibration control effects. Numerical analysis of a structure with linear damping devices such as VDs and AMD using LQR proves that the proposed method can evaluate equivalent damping ratios quite precisely. The method is also validated to be applicable for estimating equivalent damping ratios of a structure with nonlinear devices such as Coulomb-type FDs. Closed-form formulas were derived based on a probabilistic concept to obtain fundamental modal damping ratio without carrying out structural analysis. It was found through the numerical analysis that the closed-form formulas derived in this study are effective in estimating equivalent damping ratio of a structure with various damping devices operated by both linear and nonlinear control laws. The proposed method is expected to be a convenient and powerful tool for comparing control effect of various added dampers and for preliminary design of a structure with active or passive damping devices.

Acknowledgements

The work presented in this paper was partially supported by both the National Research Laboratory Program (Project No. M1-0203-00-0068) from the Ministry

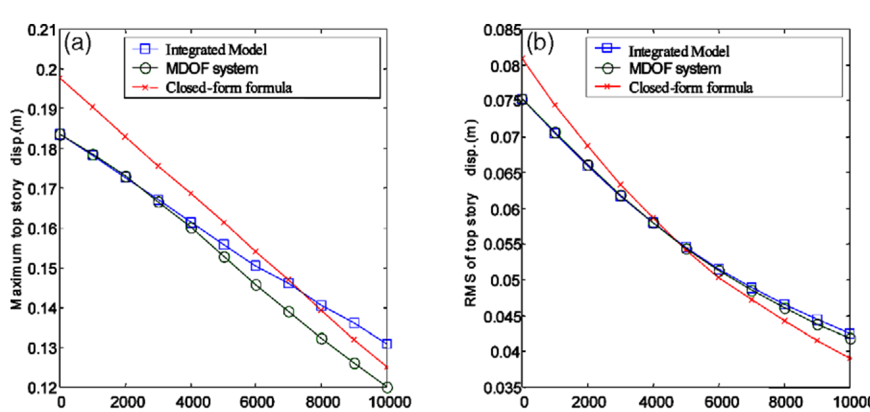

Fig. 13. Top-story displacement of the model structure with FDs: (a) maximum displacements; (b) RMS displacements.

of Science and Technology in Korea and the Korea Science and Engineering Foundation (KOSEF) through Smart Infra-Structure Technology Center (SISTeC) at Korea Advanced Institute of Science and Technology(KAIST).

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