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Estimation of the modal mass of a structure with a tuned-mass damper using H-infinity optimal model reduction

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Abstract

Accurate estimation of the modal mass of a structure is essential for analyzing the structural response under disturbance and for designing mass-type control devices used to enhance the serviceability of the structure. The modal mass of a real structure differs from that of the mathematical model due to the error introduced by the assumptions made for analysis and by the change in dynamic characteristics occurring during the construction process. In this study, a procedure is proposed for estimating the modal mass of a real structure with a tuned-mass damper based on the H-infinity optimal model reduction technique. The modal mass is obtained from the relationship between the observability–controllability matrices realized from the system identification and the typical two-degree-of-freedom state space model. The proposed method is verified through the analysis of three model structures.

Keywords: Modal mass; System identification; H-infinity optimal model reduction; Tuned-mass dampers

1. Introduction

Modal analysis that estimates the dynamic structural responses by superposing responses of a series of single-degree-of-freedom (SDOF) systems is a powerful dynamic analysis method for linear multiple-degree-of-freedom (MDOF) systems. It is especially effective for buildings and civil structures whose behaviors are governed by a few lowest modal responses. For modal analysis, the eigenvectors, which form the vector space basis of the structural response, are required. Since eigenvectors obtained from eigenvalue analysis have relative values in the modal space, they are generally normalized to a certain value. The normalization of eigenvectors is achieved by: (i) setting an arbitrary element of each vector to a unit value; (ii) setting the largest element of each vector to a unit value; or (iii) setting the modal mass of each mode to a unit value. The modal mass (or generalized mass) of each mode also has relative value depending on how the eigenvectors are normalized. For a modal mass to have physical significance, it should be combined with and characterized by other variables that have physical meaning. Consequently, the magnitude of a modal mass depends on the normalization method and physical characteristics of other variables.

The modal mass is defined for a separation of modal space that has a linearly independent basis, and its definition and analytical methods of computation are introduced in numerous literature works dealing with structural dynamics [1–3]. For structures with mass-type vibration control devices, such as tuned-mass dampers (TMDs) and active mass drivers, the accurate estimation of the modal mass of a structure is especially important because the dynamic characteristics of the devices are determined on the basis of the modal mass of the structure. The modal mass is also required for evaluation of the control effect of the device and maintaining optimal performance after the device is installed.

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The system identification methods for estimating the dynamic characteristics of a structure from shaking table tests and the recorded response time histories have been widely studied: the analytical and experimental techniques for identification of modal characteristics were addressed in [4–6] dealing with identification techniques in time and frequency domains; system identification of nonlinear systems was developed in [7] and the real-time identification methods were addressed in [8]; a computer program was developed for convenient system identification [9]; mass and stiffness matrices of shear buildings were estimated from modal test data in [10]; and output-only parameter estimation of base-excited structures was studied in [11]. Most of the previous studies have concentrated on the estimation of dynamic properties such as: transfer function, natural frequencies, damping ratios, and modal shapes; however research on the identification of the modal mass has rarely been performed.

In this paper, an identification method for the modal mass of a structure is proposed for accurate analysis and precise design of mass-type vibration control devices, such as TMDs. The modal frequency and the damping ratio are obtained first by a conventional system identification technique, and the prototype state space equation for the equivalent single-degree-of-freedom (SDOF) system is established. The modal mass is then obtained by comparing with the state space equation obtained from system identification. In order to consider the effect of output on the modal mass evaluation, separate equations for displacement and acceleration outputs are proposed. The effect of noise in the output is also considered in the formulation. It is assumed in the system identification process that both the input and the output are known even though this is impractical in the general case. The displacement and acceleration of a structure are taken as outputs. The technique is applied to three examples. The white noise is applied as an input in the first two numerical simulations and a sinusoidal force is applied between the TMD and a tower in the third example. Only the modal mass corresponding to the first mode is obtained except when the structure is vibrated at a specific frequency.

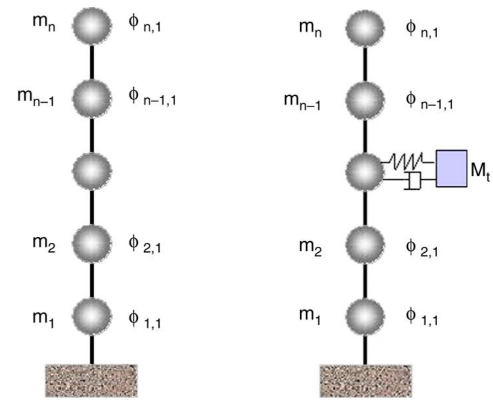
2. Computation of modal mass

2.1. Modal mass obtained from normalization of eigenvectors

The modal mass of a multiple-degree-of-freedom (MDOF) system with n discrete masses (Fig. 1(a)) corresponding to a certain mode can be estimated approximately as

$$M = \sum_{i=1}^{n} m_i \phi_i^2 \quad (1)$$

where m_i is the mass of the i-th DOF and ϕ_i is the i-th component of a certain mode shape vector (usually the first mode). The modal mass obtained above depends on how the modal vectors are determined. If the i-th component of a mode shape vector is normalized to unit value, the modal mass is obtained as



(a) n-DOF system. (b) n-DOF system with a TMD.

Fig. 1. Multiple-degree-of-freedom analysis models.

$$M_i = m_n \left(\frac{\phi_1}{\phi_i}\right)^2 + m_{n-1}\left(\frac{\phi_2}{\phi_i}\right)^2 + \cdots m_i (1)^2 \cdots + m_1 \left(\frac{\phi_n}{\phi_i}\right)^2. \quad (2)$$

When the j-th component of the mode shape vector is normalized to unit value, the modal mass is obtained as

$$M_j = m_n \left(\frac{\phi_1}{\phi_j}\right)^2 + m_{n-1}\left(\frac{\phi_2}{\phi_j}\right)^2 + \cdots m_j (1)^2 \cdots + m_1 \left(\frac{\phi_n}{\phi_j}\right)^2. \quad (3)$$

If ϕ_i^2 and ϕ_j^2 are multiplied by Eqs. (2) and (3), respectively, the following relationship holds:

$$M_i = \left(\frac{\phi_j}{\phi_i}\right)^2 M_j. \quad (4)$$

Eq. (4) demonstrates that the modal mass obtained by normalization of the i-th component of a mode shape vector is proportional to that obtained by normalization of the j-th component of the mode shape vector multiplied by the square of the relative magnitude of the j-th and the i-th components of the mode shape vector.

As can be observed in the above equations, the modal mass of a structure corresponding to a certain value can be obtained if the mass matrix and the mode shape vector are known. Previous research on system identification mostly focused on the estimation of dynamic characteristics of structures, such as natural frequencies, mode shape vectors. The modal masses are available after mode shapes are identified, which requires a lot of effort and experimental data. In this study a procedure is proposed for obtaining the modal mass directly without knowledge of the mode shape vectors.

2.2. Modal mass of a structure with a TMD

The mass of a TMD is generally determined as a portion of the modal mass of a main structure corresponding to a specific mode to which the TMD is tuned. As the mass of the damper depends on the modal mass of the structure, how the modal mass is determined is an important factor in the design of a TMD. The modal mass varies with the location of the TMD, and the variation can be predicted using Eq. (4). When a TMD is installed in an arbitrary floor of an MDOF structure (Fig. 1(b)), the equation of motion of the system, excluding the damping term for simplicity, can be constructed as

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & M_t \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}} \\ \ddot{\mathbf{x}}_t \end{bmatrix} + \begin{bmatrix} \mathbf{K} + \mathbf{S}K_t\mathbf{S}^{\mathrm{T}} & -\mathbf{S}K_t \\ -\mathbf{S}K_t & K_t \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_t \end{bmatrix} = \begin{bmatrix} \mathbf{E} \\ \mathbf{0} \end{bmatrix} \mathbf{f} \quad (5)$$

where $\mathbf{M}$ and $\mathbf{K}$ are the mass and the stiffness matrices of the structure, respectively; M_t and K_t are the mass and the stiffness of the TMD; $\mathbf{S}$ is the matrix representing the location of the TMD; and $\mathbf{x}$ and x_t are the vectors of displacements of the structure and the TMD, respectively; $\mathbf{E}$ is the unit matrix having the dimension of the structure; and $\mathbf{f}$ is the external load vector acting on nodal points of the structure. Since a TMD is usually applied to reduce the response of a specific mode, the equation of motion of the system is simplified for the specific mode as follows, assuming that the effect of other modes is negligible:

$$\begin{bmatrix} \overline{M}_1 & 0 \\ 0 & M_t \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{x}_t \end{bmatrix} + \begin{bmatrix} \overline{M}_1\omega_1^2 + K_t\phi_{t,1}^2 & -\phi_{t,1}K_t \\ -\phi_{t,1}K_t & K_t \end{bmatrix} \begin{bmatrix} \eta_1 \\ x_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{f}_1 \quad (6)$$

where $\overline{M}_1$, ω_1, $\bar{f}_1$, and η_1 are the mass, natural frequency, generalized force, and generalized displacement of the first mode, and $\phi_{t,1}$ is the mode shape value at the story where the TMD is installed.

If the modal mass of the system is defined as the mass when the component of the mode shape vector corresponding to the location of the TMD is normalized to one, then it can be noted in Eq. (6) that the dynamic property of the TMD becomes independent of the mode shape of the structure. This enables the estimation of modal mass of the structure in accordance with the location of the TMD. If the generalized modal mass of a structure obtained by normalizing the largest component of the mode vector to one is $\overline{M}_H$, the modal mass of the system when a TMD is installed, $\overline{M}_t$, can be obtained as follows using the mode shape component, ϕ_t, corresponding to the DOF associated with the location of the TMD:

$$\overline{M}_t = \left(\frac{\phi_H}{\phi_t}\right)^2 \overline{M}_H = \frac{1}{\phi_t^2}\overline{M}_H. \quad (7)$$

As the vibration control effect of a TMD generally increases with increasing ratio of the mass of the TMD to the structural modal mass, a smaller modal mass would be beneficial for vibration reduction if the mass of the device remains constant. Therefore it can be seen from Eq. (7) that the effect of the location of a TMD is significant, and that the modal mass can be used as a parameter to represent the location of the TMD.

2.3. Modal mass from the system identification method

The dynamic properties of a structure predicted by eigenvalue analysis may be different from the exact properties due mainly to the assumption associated with the analytical modeling. The precise dynamic properties can be measured by experiments after the construction of the structure is completed. The mass of a structure can be estimated fairly precisely by simple computation; however the modal mass is a dynamic property related not only to mass distribution but also to mode shape vectors. In this study the system identification technique is extended to obtain the modal mass of a structure.

System identification is a technique used to identify dynamic properties of a structure such as transfer functions. Various methods have been developed both in time and in frequency domains according to the characteristics of inputs and outputs. Also various models such as ARMA [12], OE [9], and Box–Jenkins [13] are used in accordance with the order and shape of the polynomials to represent the transfer functions of the system. The identification model can be expressed in terms of either polynomials or state equations in discrete or continuous time and frequency domains.

The system identification model used in this study to estimate the modal mass is the state space equation model in the continuous time domain [14,15]. The state equations constructed by system identification provide transfer functions containing higher modes. To obtain the modal mass for the first mode, the state equations need to be condensed into those for the single-degree-of-freedom (SDOF) system. In this study the condensation is carried out using the H-infinity optimum model reduction technique, which represents the difference between the real structure and the analysis model as H∞ Norm and minimizes it. The H-infinity reduction technique is known to be more accurate than the conventional modal condensation technique which simply truncates the higher modes [16]. In this study the higher order state equations are reduced to the second-order state equations using the program code MATLAB which adapts the H-infinity reduction technique. Fig. 2 shows the process for estimating the modal participation factor of the first mode proposed in this study. After selection of the input and output response, state equations with appropriate order are constructed through the system identification process (a) and (b); as the state equations usually have high order, they are condensed into equations with order two using the H-infinity model reduction technique (c); finally the modal mass of the first mode can be obtained by comparing the condensed state equations with those of prototype state equations (d) and (e).

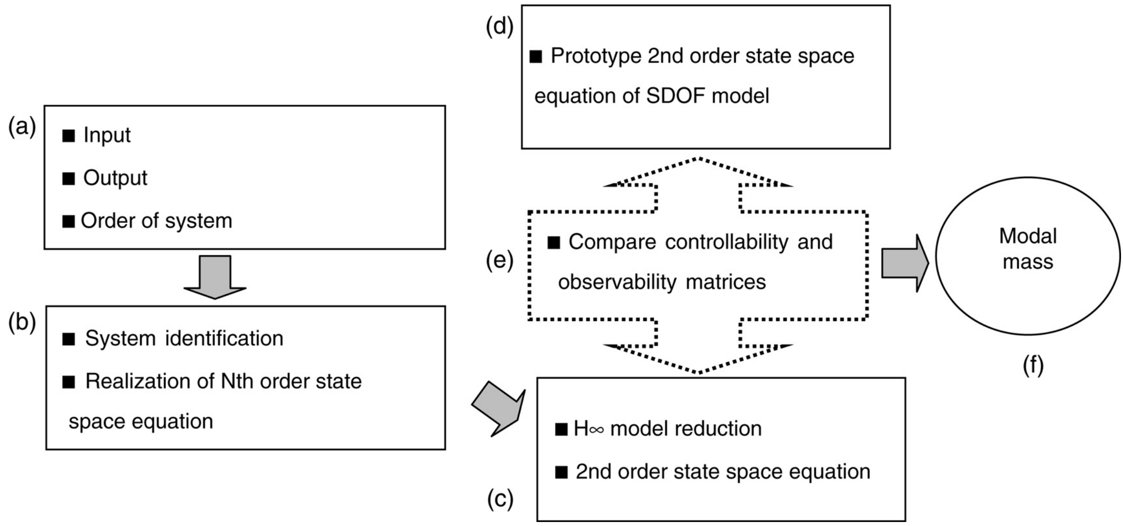


Fig. 2. Proposed process of modal mass estimation.

The state space equation of a SDOF system derived using system identification and H-infinity model reduction is presented in Eq. (8):

$$\dot{\bar{\mathbf{z}}} = \bar{\mathbf{A}}\bar{\mathbf{z}} + \bar{\mathbf{B}}\mathbf{u} \quad (8a)$$

$$\mathbf{y} = \bar{\mathbf{C}}\bar{\mathbf{z}} + \bar{\mathbf{D}}\mathbf{u} \quad (8b)$$

where $\bar{\mathbf{z}}$ is the 2×1 state variable vector; $\mathbf{u}$ and $\mathbf{y}$ are the input and output of the structure, respectively; and $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{C}}$, and $\bar{\mathbf{D}}$ are the system matrices with dimensions of 2×2, 2×1, 1×2 and 1×1, respectively, obtained from system identification. The first mode is assumed to be realized by the H-infinity optimal model reduction of the system identification model with a high order transfer function. In a state space equation such as Eq. (8), there can be an infinite number of solutions and system matrices that satisfy the relation between the input $\mathbf{u}$ and the output $\mathbf{y}$, and therefore the state variable has no physical meaning. However there exists a state equation in which the state variable retains physical meaning, which can be obtained in the process of transforming the equation of motion of a SDOF system into a state space equation. The equation of motion of a SDOF system is expressed as

$$M\ddot{x} + C\dot{x} + Kx = f \quad (9)$$

where M, C, and K are the mass, damping, and stiffness of the SDOF system; f is the external load; and x is the displacement of the system. Eq. (9) can be transformed into the state equations:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} \quad (10a)$$

$$\mathbf{y} = \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{u} \quad (10b)$$

where $z = \begin{bmatrix} x & \dot{x} \end{bmatrix}^{\mathrm{T}}$ and the matrices $\mathbf{A}$ and $\mathbf{B}$ are expressed as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -K/M & -C/M \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_1^2 & -2\xi_1\omega_1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 1/M \end{bmatrix} \quad (11)$$

where ω_1 and ξ_1 are the natural frequency and damping ratio of the system, respectively. The matrices $\mathbf{C}$ and $\mathbf{D}$ change in accordance with the type of the measured response, and they are determined for displacement and acceleration responses in this study. For the case where the displacement response is measured, those matrices are

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}. \quad (12)$$

Eq. (12) becomes as follows if the acceleration response is measured:

$$\mathbf{C} = \begin{bmatrix} -K/M & -C/M \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1/M \end{bmatrix} \quad (13)$$

The state variables in Eq. (10) have physical meaning such as displacement and velocity. Also the system matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are composed of dynamic properties such as the mass, damping, and stiffness of the structure. Therefore once the state equation obtained from system identification is transformed into a form such as Eq. (11) with system matrices having physical meaning, it is possible to derive the modal mass from the equation.

When the acceleration response is measured, the modal mass can be computed from the matrix $\mathbf{D}$ which is expressed as the inverse of the mass M as presented in Eq. (13). Also the matrix $\bar{\mathbf{D}}$ in Eq. (8) is independent of the transformation matrix, and thereby is equal to the matrix $\mathbf{D}$ in Eq. (13). Therefore, the modal mass can be obtained from the matrix $\bar{\mathbf{D}}$ constructed from system identification. In practice, however, significant error can be involved in $\bar{\mathbf{D}}$ in the process of system identification and an accurate modal mass can hardly be obtained. Therefore, a more generalized method is required for modal mass identification, which does not depend on the matrix $\bar{\mathbf{D}}$.

The state equation of Eq. (8) obtained from system identification can be transformed into prototype state equations of Eq. (10) using matrix transformation; the state variable $\bar{\mathbf{z}}$ in Eq. (8) can be transformed into $\mathbf{z}$ in Eq. (10)

using the transformation matrix $\mathbf{T}$ such that

$$\mathbf{z} = \mathbf{T}\overline{\mathbf{z}} \tag{14}$$

Substituting Eq. (14) into Eq. (10) results in

$$\dot{\overline{\mathbf{z}}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\overline{\mathbf{z}} + \mathbf{T}^{-1}\mathbf{B}\mathbf{u} \tag{15a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{T}\overline{\mathbf{z}} + \mathbf{D}\mathbf{u} \tag{15b}$$

In comparison with Eq. (8), the system matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$, $\overline{\mathbf{C}}$, and $\overline{\mathbf{D}}$ correspond to

$$\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \quad \overline{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}, \quad \overline{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1}, \quad \overline{\mathbf{D}} = \mathbf{D}. \tag{16}$$

Eq. (16) implies that a system identified in arbitrary format can be transformed into a prototype state equation with variables having physical meaning using the transformation matrix $\mathbf{T}$. In order to obtain the transformation matrix $\mathbf{T}$, Eq. (16) is modified as

$$\mathbf{A}\mathbf{T} = \mathbf{T}\overline{\mathbf{A}} \tag{17a}$$

$$\mathbf{B} = \mathbf{T}\overline{\mathbf{B}}. \tag{17b}$$

Multiplying the matrix $\mathbf{A}$ by the left-hand side of Eq. (17b) and using the relation of Eq. (17a), the following relation is obtained:

$$\mathbf{A}\mathbf{B} = \mathbf{T}\overline{\mathbf{A}}\,\overline{\mathbf{B}}. \tag{18}$$

Since Eqs. (17b) and (18) are column vectors with dimensions of 2×1, the controllability matrix can be obtained as follows by rearranging those equations to obtain the transformation matrix $\mathbf{T}$:

$$\left[\mathbf{B} \;\; \mathbf{A}\mathbf{B}\right] = \mathbf{T}\left[\overline{\mathbf{B}} \;\; \overline{\mathbf{A}}\,\overline{\mathbf{B}}\right] \tag{19}$$

Similarly, the observability matrix can be obtained as

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \end{bmatrix} \mathbf{T} = \begin{bmatrix} \overline{\mathbf{C}} \\ \overline{\mathbf{C}}\,\overline{\mathbf{A}} \end{bmatrix}. \tag{20}$$

From Eq. (19) or Eq. (20), the transformation matrix $\mathbf{T}$ can be obtained as [10]

$$\mathbf{T} = \left[\mathbf{B} \;\; \mathbf{A}\mathbf{B}\right]\left[\overline{\mathbf{B}} \;\; \overline{\mathbf{A}}\,\overline{\mathbf{B}}\right]^{-1} \tag{21a}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \end{bmatrix}^{-1} \begin{bmatrix} \overline{\mathbf{C}} \\ \overline{\mathbf{C}}\,\overline{\mathbf{A}} \end{bmatrix}. \tag{21b}$$

Even though Eqs. (21a) and (21b) represent the same transformation matrix $\mathbf{T}$, the matrix $\mathbf{B}$ in Eq. (21a) includes the unknown modal mass $\mathbf{M}$ as shown in Eq. (11), and therefore Eq. (21a) cannot be used to compute $\mathbf{T}$. In contrast, the matrices $\mathbf{C}$, $\mathbf{A}$, $\overline{\mathbf{C}}$, and $\overline{\mathbf{A}}$ in (21b) can be easily obtained from system identification, which leads to the transformation matrix $\mathbf{T}$. Substituting $\mathbf{T}$ of Eq. (21b) into Eq. (19) results in

$$\left[\mathbf{B} \;\; \mathbf{A}\mathbf{B}\right] = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \end{bmatrix}^{-1} \begin{bmatrix} \overline{\mathbf{C}} \\ \overline{\mathbf{C}}\,\overline{\mathbf{A}} \end{bmatrix} \left[\overline{\mathbf{B}} \;\; \overline{\mathbf{A}}\,\overline{\mathbf{B}}\right]. \tag{22}$$

The right-hand side of Eq. (22) is composed of matrices which can be obtained by conventional system identification, whereas the matrices in the left-hand side are available after being transformed into the state equation of a SDOF system.

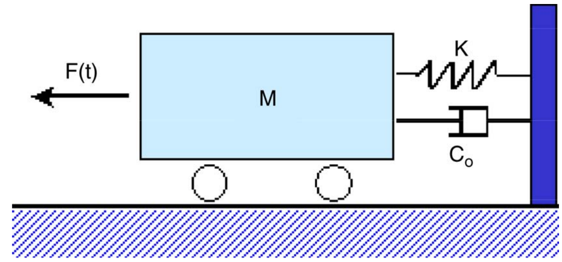


Fig. 3. A single-degree-of-freedom system subjected to a dynamic load.

The matrices $\mathbf{A}$ and $\mathbf{B}$ are rewritten here for convenience:

$$\left[\mathbf{B} \;\; \mathbf{A}\mathbf{B}\right] = \left[\mathbf{B}_o \;\; \mathbf{A}\mathbf{B}_o\right] \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \frac{1}{\overline{\overline{M}}} \tag{23}$$

where $\mathbf{B}_o = \left[0 \;\; 1\right]^{\mathrm{T}}$. Substituting Eq. (23) into Eq. (22) leads to

$$\frac{1}{\overline{\overline{M}}}\mathbf{E} = \left[\mathbf{B}_o \;\; \mathbf{A}\mathbf{B}_o\right]^{-1} \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \end{bmatrix}^{-1} \begin{bmatrix} \overline{\mathbf{C}} \\ \overline{\mathbf{C}}\,\overline{\mathbf{A}} \end{bmatrix} \left[\overline{\mathbf{B}} \;\; \overline{\mathbf{A}}\,\overline{\mathbf{B}}\right] \tag{24}$$

where $\mathbf{I}$ is the 2×2 identity matrix. In Eq. (24) the matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$, and $\overline{\mathbf{C}}$ can be obtained from system identification and the matrices $\mathbf{A}$, $\mathbf{B}_o$, and $\mathbf{C}$ can be expressed using the natural frequency and damping ratio (ω_1 and ξ_1) estimated through the system identification process. Consequently, the modal mass $\overline{M}$ is readily obtained.

3. Numerical examples

3.1. Modal mass of a SDOF system

The proposed method for estimating the modal mass is applied to a SDOF system subjected to an external excitation $F(t)$ as shown in Fig. 3. A low-pass filtered white noise is used for an input excitation. The dynamic properties of the system and the characteristics of the input excitation are presented in Table 1. The time interval of the excitation is 0.01 s and the total duration is 600 s. The system matrices ($\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$, $\overline{\mathbf{C}}$, and $\overline{\mathbf{D}}$) and the dynamic characteristics identified using the displacement and acceleration outputs are summarized in Tables 2 and 3, respectively.

It can be observed that the natural frequency and the damping ratio obtained from the system matrices are almost identical to those obtained from eigenvalue analysis. The modal mass computed using Eq. (24) for displacement and acceleration outputs matches the true mass of the system with an error of 0.26% and 2.2%, respectively. The reason for the larger error in the case of using acceleration is due to the force included in the acceleration. Then the effect of noise included in the output on the evaluation of the modal mass is also investigated. The acceleration response is used for output, and the added noise is assumed to be a white noise with its RMS amplitude increasing up to 25% of the acceleration output response.

Table 4 presents the modal mass obtained from the output response including various levels of noise. It can be noticed

Table 1
Properties of the SDOF system and the input excitation

SDOF system	Mass	1170 ton
	Natural frequency	0.7 Hz
	Damping ratio	0.5%
	System matrices	$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -19.3444 & -0.044 \end{bmatrix}$
		$\mathbf{B} = \frac{1}{M}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
		$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ (Disp.)
		$\mathbf{C} = \begin{bmatrix} -19.3444 & -0.044 \end{bmatrix}$ (Acc.)
Excitation	Type	Low-pass filtered white noise
	Amplitude	±100 when normalized to the structural mass
	Filter bandwidth	Less than 2 Hz
	Filter transfer function	$\frac{0.0036s^2 + 0.0072s + 0.0036}{1.0000s^2 - 1.8227s + 0.8372}$
Analysis condition	Time step	0.01 s
	Duration	600 s

Table 2
System identification results using displacement response

System matrices	Dynamic characteristics		
	Mass (error)	Frequency (Hz)	Damping ratio (%)
$\overline{\mathbf{A}} = \begin{bmatrix} 99.9134 & 100.0544 \\ -100.0101 & -99.9577 \end{bmatrix}$ $\overline{\mathbf{B}} = 10^{-4}\begin{bmatrix} 0.1279 \\ -0.0427 \end{bmatrix}$ $\overline{\mathbf{C}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$	1174 ton (+0.26%)	0.6998	0.504

Table 3
System identification results using acceleration response

System matrices	Dynamic characteristics		
	Mass (error)	Frequency (Hz)	Damping ratio (%)
$\overline{\mathbf{A}} = \begin{bmatrix} 99.9135 & 100.0543 \\ -100.0102 & -99.9575 \end{bmatrix}$ $\overline{\mathbf{B}} = 10^{-3}\begin{bmatrix} -0.2452 \\ 0.0818 \end{bmatrix}$ $\overline{\mathbf{C}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$	1184 ton (+2.2%)	0.6998	0.501

Table 4
Effect of noise on the modal mass estimation

Noise level (%)	Modal mass	Error
0	1184 ton	–
7	1184.8 ton	–
15	1185.8 ton	0.1%
17	1186.1 ton	0.18%
18	−30.2709	Unstable
20	−31.7509	Unstable

that up to a certain level of noise (17% in this case) the modal mass can be estimated. However for a higher level of noise the prediction of modal mass becomes unstable. It can be concluded that the magnitude of the noise influences the stability of the observability and controllability matrices, and large noise leads to an unreasonable value for the modal mass.

3.2. Modal mass of an MDOF system

The MDOF system used in the numerical analysis is the five-story shear building. The fundamental modal mass of 1170 ton was obtained using the fundamental mode shape vector normalized in such a way that the top-floor component is one. The dynamic characteristics of the structure and the input excitation are summarized in Table 5. The input excitation, which is the same white noise time history as was used in the previous example, is assumed to act on the top floor of the structure.

Numerical analysis has been carried out to compute the displacement and the acceleration at the top story. With the input excitation and the output responses, a state space equation model of order 10 is used for system identification. The results are transformed into the state equation of order 2 using the H-infinity optimal model reduction method to obtain the first modal mass. Table 6 presents the condensed model and the modal mass obtained using the top-floor displacement and acceleration responses. As in the previous example, the error associated with the modal mass is larger when the acceleration is used in the system identification. This is due to the fact that the contribution of higher modes plays a role similar to that of noise when the acceleration is used.

3.3. Modal mass of structures with TMD

In this section the modal mass of a structure with a TMD is evaluated and the variation of the modal mass with varying TMD location is investigated. To vibrate the structure, the TMD is excited first with a forcing frequency equal to the tuned natural frequency. As the TMD is tuned to the natural frequency of the structure, even a small movement of the TMD can easily excite the structure.

If it is assumed that the structure is excited only in the tuned mode and that the contribution of the other modes is negligible, the equation of motion in Eq. (6) can be simplified as

$$\overline{M}_1\ddot{\eta}_1 + \overline{M}_1\omega_1^2\eta_1 + K_t\eta_1 - K_t x_t = -U_r \tag{25a}$$

$$M_t\ddot{x}_t + K_t x_t - K_t\eta_1 = U_r \tag{25b}$$

where U_r is the force acting on the TMD, which also acts on the structure as a reaction. In Eq. (25) it is assumed that no external load is exerted on the structure and the mode shape vector is normalized in such a way that the component of the modal vector corresponding to the top floor becomes unit value. Since the reaction force U_r is hard to measure, it would be more convenient to express the equation without the forcing term. Substituting U_r in Eq. (25b) into (25a) and adding the damping term results in

$$\overline{M}_1\ddot{\eta}_1 + 2\xi_1\omega_1\overline{M}_1 + \overline{M}_1\omega_1^2\eta_1 = -m_t\ddot{x}_t = u. \tag{26}$$

Table 5
MDOF system and input excitation

Story mass Story damping Story stiffness	416.84 ton 227,270 N s/m 100 MN/m					
	Mode	1st	2nd	3rd	4th	5th
Modal characteristics	Mass (ton)	1170	1385	2007	3992	14,442
	Damping (%)	0.5	1.46	2.31	2.96	3.38
	Frequency (Hz)	0.701	2.048	3.228	4.147	4.731
Response output Input excitation	Disp. and acc. of top floor Same as Ex. 1					

In Eq. (26), it can be observed that the inertia force of the TMD excites the structure. As the TMD mass m_t can be precisely estimated, the input u required for system identification can be obtained by measuring the acceleration of the TMD. After the optimally reduced state equation is obtained using the displacement or acceleration response of the floor with the TMD, the modal mass can be estimated using Eq. (24).

In order to investigate the effect of the location of a TMD on the modal mass estimation, a five-story shear building with a TMD is analyzed first. The mass of the TMD is assumed to be 11.7 ton which is 1% of the modal mass of the first mode, which is 1170 ton. A harmonic excitation force with its frequency equal to the tuned natural frequency acting between the TMD and the structure is used as an input. Table 7 compares the identified modal masses and those obtained from eigenvalue analysis, where it can be observed that they agree quite well. It can also be observed that the modal mass increases as the location of the TMD moves down, which results in a decrease in the mass ratio of the TMD and the generalized mass. This implies that, as expected, the effectiveness of the TMD changes depending on the TMD location, and that the modal mass can be used as a parameter for measuring the effectiveness of the TMD qualitatively.

Table 6
Condensed model and modal mass estimation results

Response output	Condensed system matrices	Fundamental modal mass (error)
Top floor disp.	$\overline{\mathbf{A}} = \begin{bmatrix} -0.0223 & 4.4085 \\ -4.4085 & -0.0219 \end{bmatrix}$ $\overline{\mathbf{B}} = \begin{bmatrix} -0.0099 \\ -0.0097 \end{bmatrix}$ $\overline{\mathbf{C}} = \begin{bmatrix} -0.0099 & 0.0097 \end{bmatrix}$	1178.1 ton (+0.69%)
Top floor acc.	$\overline{\mathbf{A}} = \begin{bmatrix} -0.0405 & 4.4085 \\ -4.4085 & -0.0037 \end{bmatrix}$ $\overline{\mathbf{B}} = \begin{bmatrix} -0.0743 \\ -0.0225 \end{bmatrix}$ $\overline{\mathbf{C}} = \begin{bmatrix} 0.0743 & -0.0225 \end{bmatrix}$	1302.2 ton (+11.3%)

Fig. 4. Control tower of the Yang-Yang International Airport, Korea.

Next, the procedure for estimating modal mass is applied to a real structure with a TMD. The structure, shown in Fig. 4, is a control tower of the Yang-Yang International Airport located in the eastern part of Korea. The height of the tower is 80.1 m and the TMD is installed at the height of 72 m from the ground. The dynamic characteristics of the structure computed numerically in the design stage are presented in Table 8 along with the properties of the TMD. Right after the construction of the structure was completed, experiments were carried out to evaluate the effectiveness of the TMD; at first the TMD was excited manually, and subsequently the inertia force of the moving mass excited the tower. During the experiment, the acceleration of the tower and the TMD were measured, and the relative displacement between the tower and the TMD was also recorded. Fig. 5 presents the acceleration time history of the tower and the TMD used in the system identification. Table 9 summarizes the results of the system identification and the estimated modal mass. From comparison with Table 8, it can be observed that the dynamic characteristics of the structure obtained from system identification are somewhat different from those obtained from eigenvalue analysis. In the case of modal mass, the difference is about 8%.

Table 7
System matrices and modal mass of a structure with a TMD

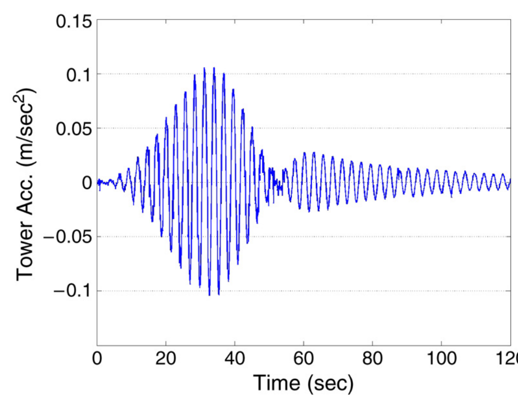
TMD location (floor)	System matrices	Modal mass (ton)	Estimated modal mass (error)
1st	$\overline{\mathbf{A}} = \begin{bmatrix} -14.6957 & -0.3510 \\ 668.7754 & 14.6515 \end{bmatrix}$ $\overline{\mathbf{B}} = \begin{bmatrix} 0.0057 \\ 0.0348 \end{bmatrix}$ $\overline{\mathbf{C}} = \begin{bmatrix} 0.0738 & -0.0019 \end{bmatrix}$	14,442	14,450 (0.06%)
3rd	$\overline{\mathbf{A}} = \begin{bmatrix} -1.8359 & -2.8344 \\ 8.0179 & 1.7918 \end{bmatrix}$ $\overline{\mathbf{B}} = \begin{bmatrix} 0.0057 \\ 0.0348 \end{bmatrix}$ $\overline{\mathbf{C}} = \begin{bmatrix} 0.0752 & -0.0134 \end{bmatrix}$	2007	2013 (0.3%)
5th	$\overline{\mathbf{A}} = \begin{bmatrix} -3.4714 & 3.7772 \\ -8.2943 & 3.4273 \end{bmatrix}$ $\overline{\mathbf{B}} = \begin{bmatrix} -0.0057 \\ -0.0424 \end{bmatrix}$ $\overline{\mathbf{C}} = \begin{bmatrix} 0.1277 & -0.0137 \end{bmatrix}$	1170	1171.3 (0.11%)

Table 8
Design parameters for the control tower and TMD

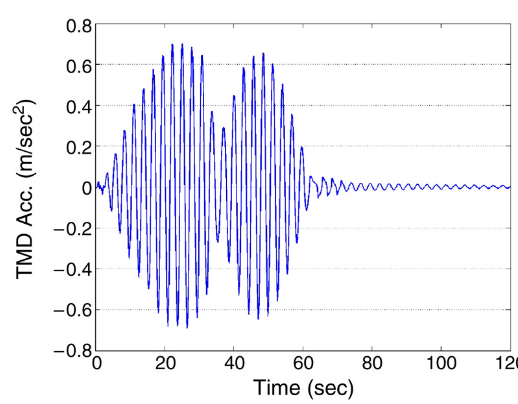
Control tower	TMD
Height: 80.1 m	Size: 2.5×2×1.8 m
Maximum width: 21.0 m	Total mass: 18 ton
Mass: 5200 ton	Moving mass: 15 ton
Modal mass: 1626.1 ton	Stroke: ±0.3 m
Natural frequency: 0.386 Hz	Natural frequency: 0.381 Hz
Damping ratio: 2% (assumed)	Damping ratio: 6.8%

Table 9
System identification of the control tower

System matrices	Dynamic characteristics		
	Mass (error)	Frequency (error)	Damping ratio
$\overline{\mathbf{A}} = \begin{bmatrix} -0.0271 & -2.2928 \\ 2.2928 & -0.0247 \end{bmatrix}$ $\overline{\mathbf{B}} = 10^{-3} \begin{bmatrix} 0.8310 \\ -0.7847 \end{bmatrix}$ $\overline{\mathbf{C}} = 10^{-3} \begin{bmatrix} -0.8310 & -0.7847 \end{bmatrix}$	1756 ton (+8%)	0.3645 Hz (−5.6%)	1.13%



(a) Acceleration of the tower.



(b) Acceleration of the TMD.

Fig. 5. Response acceleration time histories.

4. Conclusions

In this study a method was proposed for estimating the modal mass of a structure using the system identification technique. The scope of the study was limited to building-like structures in which mass is lumped for each story. The rotatory inertia mass was not considered. As the study remained at its beginning stage of formulating only the first mode of vibration, further research would be required for application to more complicated structures.

To validate the applicability of the proposed method, numerical analyses were carried out with a SDOF and five-story MDOF structures. The method was also applied to estimate the modal mass of a real structure with a TMD. The numerical analysis results showed that the modal mass estimated by the proposed method was consistent with that obtained from eigenvalue analysis. The effect of noise in the output was not significant if the noise was not large, while the estimated modal mass became unstable when the noise was larger than a certain level. The numerical examples of MDOF structures with a TMD showed that the modal mass can be used as a parameter to determine the optimal location of the TMD and the optimal design parameter of the TMD. Finally the analysis of and experiment on a control tower with a TMD showed that the modal mass of a structure with a TMD could be estimated using the acceleration responses of the structure and the TMD obtained from a vibration test.

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