Adaptive Newton-Raphson Method for Analysis of Structures with Material Nonlinearity Using Stiffness-Equivalent Load

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Abstract: This paper presents a new material nonlinear analysis algorithm which utilizes equivalent load for stiffness. The proposed algorithm requires calculation of the inverse of the global stiffness matrix only once regardless of the loading steps and provides the same results as those by the conventional method. The efficiency of the proposed algorithm depends on the ratio of the boundary DOFs - the DOFs directly connected to the elements where stiffness changed - to the total DOFs. If the number of boundary DOFs is less than about 1/8 of the band width of the global stiffness matrix the proposed algorithm is more efficient than the conventional method. The analysis results of the model structure showed that the linear analysis of a structure using the proposed algorithm produced identical responses as those by conventional plastic hinge analysis method.

Key words: material nonlinearity, stiffness-equivalent load, Newton-Raphson method, boundary DOF.

1. INTRODUCTION

Material nonlinear analysis is used when the stiffness of a structure changes due to yielding of its members; however nonlinear analysis procedure has not been widely adopted in practice because of the large computational efforts and technical difficulties involved in the modeling and analysis procedures. However the need for nonlinear analysis has been increased since the advent of performance-based seismic engineering of structures. A lot of researches have been conducted to enhance the computational efficiency and reliability of nonlinear analysis (Chen *et al.* 2004; Chen *et al.* 2004; Deng *et al.* 2001; Fafitis 2005; Huang *et al.* 2000; Kirsch *et al.* 1997; Makode *et al.* 1999; Wu *et al.* 2001). Richard *et al.* (2000) proposed an improved scheme for the nonlinear plastic hinge analysis procedure of three-dimensional structures. Iu *et al.* (2009) explained geometric and material nonlinear behavior of composite steel-concrete frame structure using refined plastic hinge method, and Audenaert *et al.* (2008) proposed an analysis procedure for an arch bridge using an elasto-plastic material model.

The basic concept of nonlinear analysis using matrix analysis or finite element method is to divide total load into many steps, carry out linear analysis in each loading step, and sum the results to obtain final responses. If the load is divided into the more loading steps, the lesser will be the error involved in the accumulation of results of linear analysis. However the number of loading steps is determined based not only on accuracy but also on computational efficiency. Therefore the response of a structure with material nonlinearity is obtained through iterative process in such a way that the out of balance load falls within a given tolerance range (Bathe 1996).

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A series of repeated linear analysis to obtain material nonlinear response requires a lot of computational effort and time; in each loading step a stiffness matrix and its inverse matrix are formed. In forming inverse matrix, computations of the third power of total degrees of freedoms (DOFs) are required. Sometimes the analysis results may not be reliable due to the accumulation of errors.

This study proposes a new algorithm for analysis of discrete structures, such as beams, columns, and braces, with material nonlinearities which computes global stiffness matrix only once regardless of the loading steps. The proposed algorithm does not require sequential application of load or the repeated compensation of out of balance load, and also capable of simulating the down slope of the stress-strain curve of concrete and the behavior after buckling of a member. On the other hand, the load-displacement relationship of an element or the constitutive relationship of a material needs to be idealized as a series of linear relations.

The key point of the proposed algorithm is to convert the stiffness change of a structure in a loading step into an equivalent load and impose it to the initial linear structure. Such an equivalent load was termed as the equivalent load for stiffness (ELS) (Kim *et al.* 2008). The global responses after change in stiffness can be calculated by simple back-substitution to the inverse matrix of the original global stiffness matrix, which has already been factorized in the initial analysis without reinversion of it. In comparison with the analysis using the conventional plastic hinge method, the proposed method provides the same results, except that the computation time may be different depending on the number of elements undergoing inelastic deformation. The proposed method is effective when only a small number of the structural members are subjected to inelastic deformation. When most of the elements undergo nonlinear behavior, the proposed algorithm requires more time for computation than conventional procedure. On the other hand, the amount of operation increases as the number of nonlinear elements increases, and in structures in which most structural members undergo inelastic deformation, the proposed method requires two to three times more operations than conventional methods. The algorithm developed in this paper is also limited to discrete members such as beams and braces and to the analysis of structures with material nonlinearities.

In this paper the basic concept of the ELS is explained first using a single-degree-of-freedom (SDOF) spring-mass system. Then an algorithm is proposed to obtain ELS for stiffness change in a multi-degrees-of-freedom (MDOF) structure. The validity of the proposed algorithm is confirmed by comparing analysis results of a braced three-story structure with those obtained from a conventional method.

2. BASIC CONCEPT OF EQUIVALENT LOAD FOR STIFFNESS (ELS)

In this section the concept of ELS is explained first and the possibility of plastic analysis using ELS is investigated. Figure 1 shows the SDOF system with a linear spring A with stiffness of 2 N/mm and a bilinear spring B with initial stiffness of 2 N/mm and post-yield

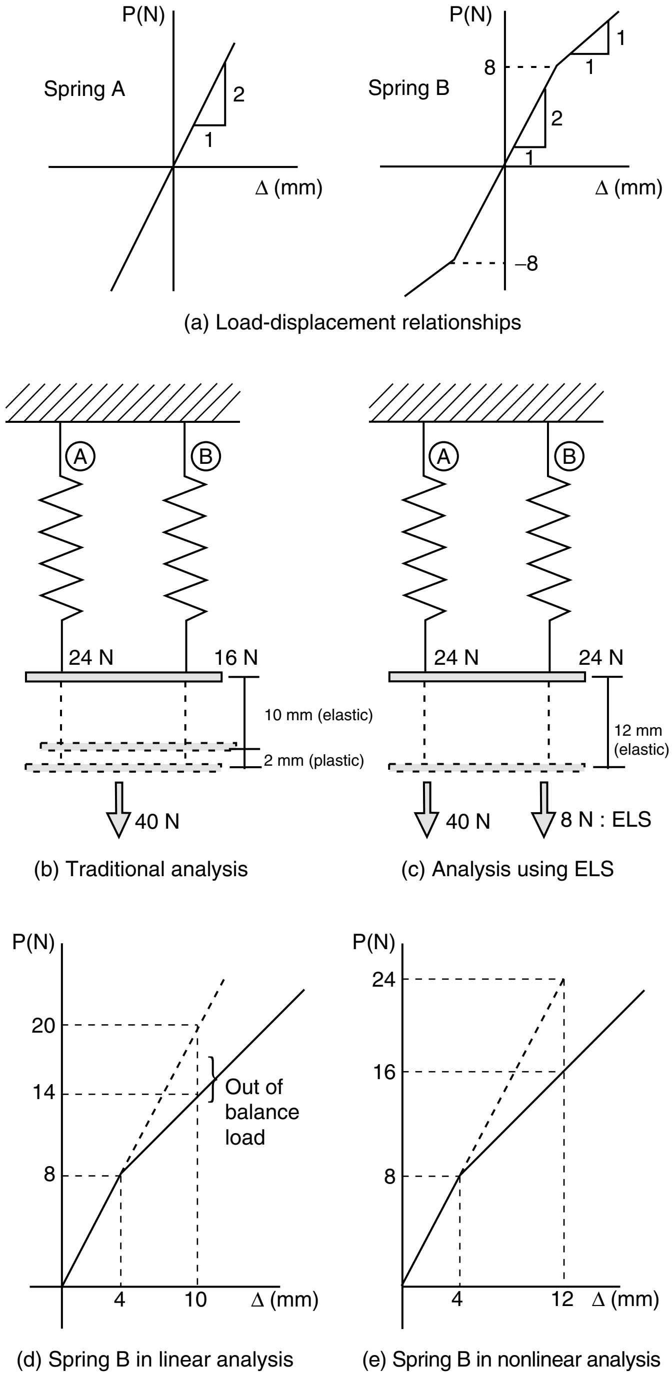


Figure 1. Equivalent load for stiffness of a SDOF system

stiffness of 1N/mm. When a vertical load of 40 N is applied, 10 mm elastic and additional 2 mm inelastic displacements occur [Figure 1(b)] and 6N of out of balance load occurs in spring B as can be seen in Figure 1(d). Repeated compensation of this out of balance load leads to additional plastic displacement of 2 mm. At this moment the forces in springs A and B are 24N and 16N, respectively [Figure 1(e)]. On the other hand, Figure 1(c) depicts that the same displacement can be obtained when the system is analyzed using the initial stiffness with additional load of 8N. In this case the force in the elastic spring A is computed directly and that of the inelastic spring B can be obtained from the force-displacement relationship shown in Figure 1(e) using the final displacement.

The above example shows that if proper additional load is added the displacement of a nonlinear structure can be computed using initial stiffness without updating the stiffness matrix or repeating compensation of out of balance load. The additional load is obtained in such a way that the effect of stiffness change on displacement is properly considered. In Figure 1(c) the additional load of 8N is the equivalent load for stiffness for the given structure and loading condition.

For a plastic analysis using ELS to be effective, ELS needs to be easily computed and the analysis procedure should be simpler than that of the conventional plastic analysis method.

The procedure to compute ELS of 8N in the structure shown in Figure 1 is summarized as follows. First, linear analysis using initial stiffness and given load produces initial displacement of 10 mm. As can be observed in Figure 1(d) the force in the spring B at the displacement of 10 mm is computed to be 20N for initial stiffness, but 14N when considered elasto-plastic behavior. The out of balance load of 6N is redistributed to the stiffness still remaining after stiffness change (it will be called 'retained stiffness' from now on). The same results can be reproduced in the same structure with initial stiffness as follows:

1) Stiffness of initial structure : 4 N/mm
2) Retained stiffness after change in stiffness : 3 N/mm
3) The out of balance load redistributed to the retained stiffness : 6 N
4) Distribution ratio of nodal load with respect to the retained stiffness : 3/4
5) The nodal load R required to distribute the out of balance load of 6 N : $R \times 3/4 = 6N$; $\therefore$ $R = 8$ N.

If the nodal load of 8 N is applied to the initial structure, the out of balance load of 6N caused by the nonlinear behavior of spring B can be redistributed to the retained stiffness. In this way the phenomenon of load redistribution caused by nonlinear behavior of members can be simulated by linear analysis of initial structure. In this example the nodal load of 8N is the ELS for the reduced stiffness after yielding of spring B.

Figure 2 shows a SDOF system composed of four springs with different nonlinearities. Also shown are the loading history and the load-displacement relationships of each spring. Table 1 presents the analysis results and the calculation process of the structure to the first loading step of 1,200N of Figure 2(b). The analysis process is summarized as follows:

1) Initial elastic analysis
   As a result of the elastic analysis using the initial stiffness of 8N/mm, elastic displacement of 150 mm and member forces of each spring were obtained.
2) Computation of the first ELS
   The stiffness and the out of balance load of each spring at the displacement of 150 mm were obtained using the load-displacement relationship of Figure 2(c). As a result, in springs B and C inelastic deformations were occurred, and unbalanced load of 250 N was occurred (200N and 50N in springs B and C, respectively). The ELS required to be imposed on the initial structure to simulate the redistribution of the out of balance load of 250N to the retained stiffness of 5N/mm is computed as follows:

$$ELS \times (5/8) = 250N, \quad \therefore ELS = 400N \tag{1}$$

By applying the ELS of 400N to the initial structure, the additional displacement of 50 mm and the additional member forces allocated to the retained stiffness of each spring are computed.

3) Computation of the 2nd ELS
   Due to the displacement and load redistribution by the application of the first ELS, the stiffness of the spring D changes from 2N/mm to −2N/mm and the ELS of 1,600N can be obtained by following the above process. If the second ELS is applied to the initial structure, the additional displacement of 200 mm and the additional member forces allocated to the retained stiffness of each spring are computed.
4) Computation of the 3rd ELS
   As a result of the displacement and load redistribution by the application of the 2nd ELS, the stiffness of the spring D changes from −2N/mm to 0N/mm and the ELS of −533.3N is obtained. By applying the third ELS of −533.3N to the initial structure, the additional displacement of −66.7 mm and the additional member forces allocated to the retained stiffness of each spring are computed.

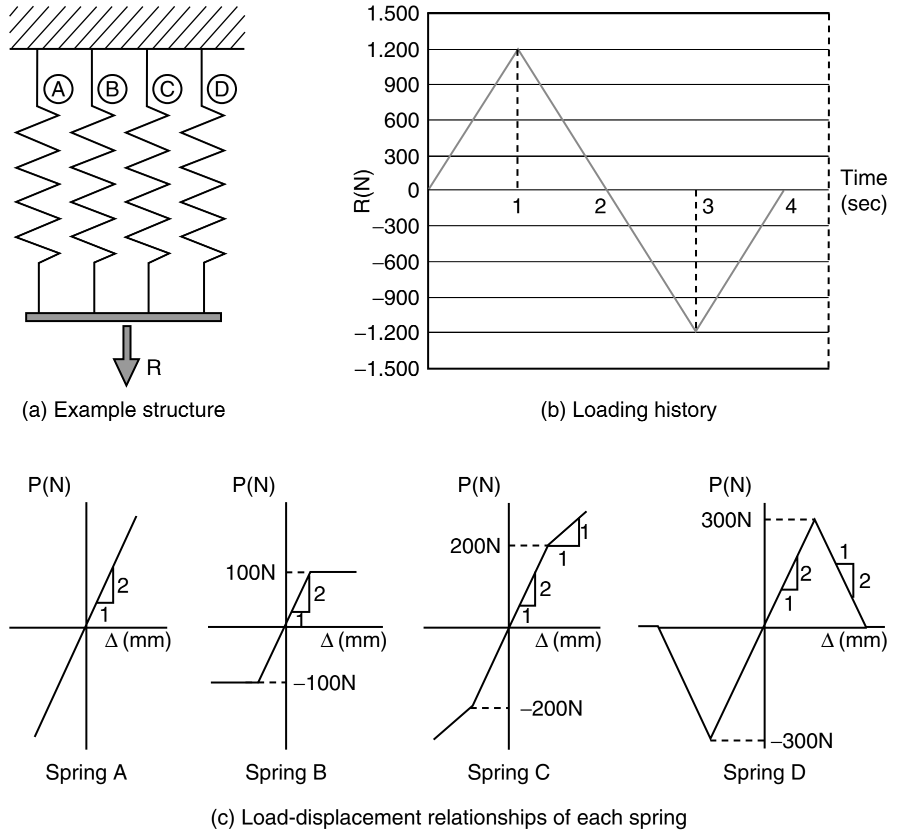


Figure 2. Example of nonlinear analysis of a SDOF system

Table 1. Analysis procedures and results

Analysis step	Detailed analysis procedure	Spring A	Spring B	Spring C	Spring D	Sum
Initial linear analysis	Initial stiffness (N/mm)	2	2	2	2	8
	Load (N)					1200
	Elastic displacement (mm)					150
	Elastic member force (N)$^{(A0)}$	300	300	300	300	1200
Computation of the 1st ELS	Stiffness at current state (N/mm)	2	0	1	2	5
	Out of balance load (N)$^{(B)}$	0	200	50	0	250
	ELS (N)		ELS × 5/8 = 250			400
	Additional displacement by ELS (mm)					50
	Member force distributed to the retained stiffness$^{(C)}$	100	0	50	100	250
	Member force = (A0) −(B) + (C)$^{(A1)}$	400	100	300	400	1200
Computation of the 2nd ELS	Stiffness at current state (N/mm)	2	0	1	−2	1
	Out of balance load (N)$^{(B)}$	0	0	0	200	200
	ELS (N)		ELS × 1/8 = 200			1600
	Additional displacement by ELS (mm)					200
	Member force distributed to the retained stiffness$^{(C)}$	400	0	200	−400	200
	Member force = (A1) −(B) + (C)$^{(A2)}$	800	100	500	−200	1200
Computation of the 3rd ELS	Stiffness at current state (N/mm)	2	0	1	0	3
	Out of balance load (N)$^{(B)}$	0	0	0	−200	−200
	ELS (N)		ELS × 3/8 = −200			−533.3
	Additional displacement by ELS (mm)					−66.7
	Member force distributed to the retained stiffness$^{(C)}$	−133.4	0	−66.7	0	−200
	Member force = (A2) −(B) + (C)	666.7	100	433.3	0	1200
Final ELS and displacement	Summation of ELS (N)		400 + 1600 − 533.3 = 1466.7			1466.7
	Final displacement (mm)		(1200 + 1466.7)/8 = 333.3			333.3

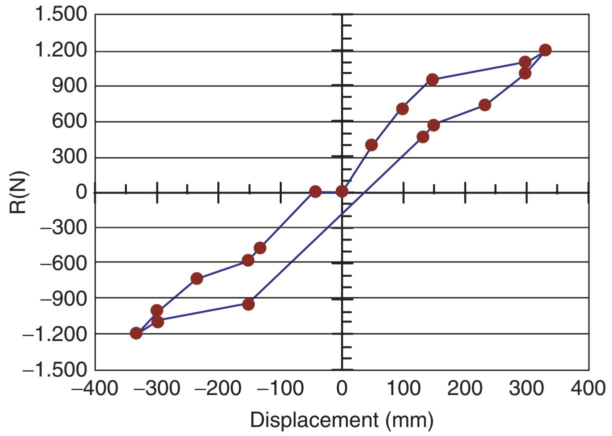


Figure 3. Load-displacement relationship of model structure

5) Final ELS and displacement
Finally, the elasto-plastic response of the structure can be reproduced by imposing both the initial load and the sum of the ELS of each step (400 + 1.600–533.3 = 1,466.7N) on the initial structure and performing linear analysis. The final displacement of the analysis becomes 333.3 mm. Figure 3 shows analysis results of the model structure in each loading step shown in Figure 2(b).

From the ELS based material nonlinear analysis of the example structures the following observations can be made:

1) Analysis of inelastic systems with material nonlinearity can be carried out with the initial elastic structure using the equivalent load for the changed stiffness.
2) The ELS is computed using the ratio of the initial stiffness to the retained stiffness after change in stiffness.
3) The efficiency in the computation of ELS in MDOF system depends on the computation of the stiffness ratio before and after yielding or of the distribution ratio of nodal load between the deleted stiffness and the retained stiffness.

3. ALGORITHM FOR MATERIAL NONLINEAR ANALYSIS

3.1. Concept of Incremental Elasto-Plastic Analysis

The equilibrium equation of a structure in finite element analysis is expressed as follows:

$$^{t}R - {}^{t}F = 0 \tag{2}$$

where, ^{t}R is the nodal load vector at time t and ^{t}F is the member end force vector. The step-by-step nonlinear analysis technique with incremental load computes the solution at time $t + \Delta t$ using the known solution at time t, where Δt is the time increment appropriate for nonlinear analysis or simply a load increment in static analysis. The equilibrium equation at time $t + \Delta t$ can be expressed as

$$^{t+\Delta t}R - {}^{t+\Delta t}F = 0 \tag{3}$$

where, $^{t+\Delta t}R$ is independent of resultant displacement while $^{t+\Delta t}F$ is dependent on the displacement and can be written as follows:

$$^{t+\Delta t}F = {}^{t}F + \Delta F \tag{4}$$

where, ΔF is the increment of the member end force vector at time $t + \Delta t$, which can be approximated using the tangent stiffness matrix ^{t}K at time t as follows:

$$\Delta F = {}^{t}K \Delta U \tag{5}$$

where, ΔU is the increment of nodal displacement vector, ^{t}K is the derivative of the member end force vector ^{t}F with respect to the displacement ^{t}U at time t:

$$^{t}K = \frac{\partial\, ^{t}F}{\partial\, ^{t}U} \tag{6}$$

If Eqns 4 – 5are substituted to Eqn 3, the following relationship can be obtained:

$$^{t}K \Delta U = {}^{t+\Delta t}R - {}^{t}F \tag{7}$$

Using the increment ΔU obtained in Eqn 7 the displacement vector at time $t + \Delta t$ can be approximated as follows

$$^{t+\Delta t}U = {}^{t}U + \Delta U \tag{8}$$

Originally the vector $^{t+\Delta t}U$ is the displacement corresponding to the load vector $^{t+\Delta t}R$ at time $t + \Delta t$. However the displacement obtained in Eqn 8 is the approximate solution based on Eqn 5 and significant error can be involved in it depending on the time or loading steps. Therefore repeated compensation of error in Eqn 2 is required and the most widely used method is the Newton-Raphson method as follows:

$$^{t+\Delta t}K^{(i-1)} \Delta U^{(i)} = {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)} \tag{9}$$

$$^{t+\Delta t}U^{(i)} = {}^{t+\Delta t}U^{(i-1)} + \Delta U^{(i)} \tag{10}$$

where, $^{t+\Delta t}U^{(0)} = {}^{t}U$; $^{t+\Delta t}K^{(0)} = {}^{t}K$; $^{t+\Delta t}F^{(0)} = {}^{t}F$, $i = 1, 2, 3, \ldots$

The above compensation is repeated until the out of balance load vector $^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)}$ becomes less than a given convergence criterion.

Newton-Raphson method includes time-consuming process of formation of tangent stiffness matrix and its inversion as can be observed in Eqn 9. Sometimes the modified Newton-Raphson method using initial tangent stiffness matrix or the quasi-Newton method using secant stiffness is employed to enhance computational efficiency. However these methods may have limitations such as decrease in accuracy when the convergence criterion is not appropriate or significant decrease in efficiency due to too many iterations. When local stiffness degradation occurs at the buckling of some members, the analysis can be stopped due to the divergence of the out of balance load.

3.2. Adaptive Substructuring Technique

The out of balance load vector ${}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)}$ has non zero values in the DOFs corresponding to the elements that modified during the repeated compensation (the DOFs directly connected to the members of which the element stiffness are changed and they will be referred to as a 'boundary DOF' from now on). In the Newton-Raphson method the tangent stiffness matrix is constructed in each step, while in the proposed method the initial stiffness matrix is used throughout the analysis and all the DOFs corresponding to the elements undergoing inelastic deformation becomes boundary DOFs.

Eqn 11 shows the equilibrium equation condensed using only the boundary DOFs:

$$\bar{K}U_a = \bar{R}_a \tag{11}$$

where, $\bar{K}$ is the stiffness matrix condensed to the boundary DOFs, U_a is the displacement vector corresponding to the boundary DOFs, $\bar{R}_a$ is the load vector condensed to the boundary DOFs. In this case the condensed stiffness matrix is composed of the stiffness matrix of linear members condensed to the boundary DOFs, $\bar{K}^u$, and the stiffness matrix of elements subjected to nonlinear deformation, K^c:

$$\bar{K} = \bar{K}^u + K^c \tag{12}$$

The stiffness matrix of nonlinear elements can be obtained as summation of element stiffness matrix $K_{(i)}$:

$$K^c = \sum_{i=1}^{M_c} K_{(i)} \tag{13}$$

where, M_c is the number of elements with inelastic deformation. As can be observed in Eqn 11 the number of equations can be reduced from that of the total DOFs to that of the boundary DOFs by condensing the equilibrium equations. However in the elasto-plastic analysis the locations of nonlinear members and the resultant boundary DOFs vary; therefore for efficient analysis the condensed matrix needs to be easily obtained even when the boundary DOFs change.

The adaptive substructuring technique has advantage in that the condensed stiffness matrix can be obtained efficiently using the inverse matrix of initial stiffness matrix when the boundary DOFs vary. The procedure of computing condensed stiffness matrix using the adaptive substructuring is as follows. First, the flexibility matrix corresponding to the boundary DOFs, F_{aa}, is computed using the unit load method:

$$\begin{bmatrix} K_{aa} & K_{ac} \\ K_{ca} & K_{cc} \end{bmatrix} \begin{bmatrix} F_{aa} \\ F_{ca} \end{bmatrix} = \begin{bmatrix} I_{aa} \\ 0 \end{bmatrix} \tag{14}$$

where, the subscript a and c represents the boundary DOF and the condensed DOF (inner DOFs of linear part), respectively, and I_{aa} is a unit matrix composed of unit vectors with unit load at each boundary DOF. The flexibility matrix corresponding to the boundary DOFs, F_{aa}, can be computed by back-substitution using the inverse matrix of the initial stiffness matrix. The condensed stiffness matrix corresponding to the boundary DOFs, $\bar{K}$, can be obtained by inverting the flexibility matrix F_{aa} as follows:

$$\bar{K} = F_{aa}^{-1} \tag{15}$$

The substructuring technique by condensing stiffness matrix with boundary DOFs is called the adaptive substructuring technique, which is composed of the process of back-substitution to obtain F_{aa} in Eqn 14 and computation of inverse matrix F_{aa} in Eqn 15. By substituting $\bar{K}$ obtained in Eqn 15 into Eqn 12, the condensed stiffness matrix of the linear part of the structure, $\bar{K}^u$, can be obtained as follows:

$$\bar{K}^u = \bar{K} - K^c \tag{16}$$

3.3. Computation of Condensed Tangent Stiffness Matrix and Equivalent Load for Stiffness

As mentioned above the Newton-Raphson method requires computation of tangent stiffness matrix and its inverse matrix in each repeated compensation, which results in a lot of computation time. In this study the condensed tangent stiffness matrix and the ELS are computed using the adaptive substructuring technique, and the adaptive Newton-Raphson method is proposed which enhances the efficiency of conventional Newton-Raphson method of Eqn 9 significantly.

Static condensation of Eqn 9 using the adaptive substructuring technique leads to:

$${}^{t+\Delta t}\bar{K}^{(i-1)} \Delta U_a^{(i)} = {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)} \tag{17}$$

where, ${}^{t+\Delta t}\bar{K}^{(i-1)}$ is the tangent stiffness matrix condensed to the boundary DOFs, and $\Delta U_a^{(i)}$ is the compensation displacement vector corresponding to the boundary DOFs. The out of balance load vector in the right hand side of Eqn 17 is also composed of elements corresponding to the boundary DOFs. The condensed tangent stiffness matrix can be obtained efficiently using the adaptive substructuring technique presented in Eqns 14–16; i.e. the flexibility matrix for boundary DOFs is obtained by inverse of initial stiffness matrix using Eqn 14 and the initial stiffness matrix ${}^0\bar{K}$ condensed to the boundary DOFs can be computed by inverse of the flexibility matrix as shown in Eqn 15. From this the initial stiffness matrix is obtained and after subtracting the initial stiffness matrix of members with inelastic deformation as in Eqn 16 the initial stiffness matrix of linear members condensed to the boundary DOFs can be obtained as follows:

$$ {}^{t+\Delta t}\bar{K}_L^{(i-1)} = {}^0\bar{K} - \sum {}^0K_{(m)} \tag{18} $$

where, the subscript m denotes the elements in nonlinear behavior, and ${}^0K_{(m)}$ is the stiffness matrix of the mth element with nonlinear deformation. Finally the condensed tangent stiffness matrix can be obtained by adding the condensed stiffness matrix of linear members and the summation of the condensed tangent stiffness matrices of nonlinear elements as follows:

$$ {}^{t+\Delta t}\bar{K}^{(i-1)} = {}^{t+\Delta t}\bar{K}_L^{(i-1)} + \sum {}^{t+\Delta t}K_{(m)}^{(i-1)} \tag{19} $$

By substituting the condensed tangent stiffness matrix into Eqn 17, the compensated displacement vector $\Delta U_a^{(i)}$ for boundary DOFs corresponding to the out of balance load vector ${}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)}$ can be obtained. On the other hand, Eqn 17 can be rewritten as follows using the ELS:

$$ {}^0\bar{K}\Delta U_a^{(i)} = {}^{t+\Delta t}Q^{(i-1)} \tag{20} $$

where, Q is the ELS required to generate the compensated displacement $\Delta U_a^{(i)}$ for boundary DOFs in the initial structure. Substituting $\Delta U_a^{(i)}$ obtained in Eqn 17 into Eqn 20 leads to the ELS as follows:

$$ {}^{t+\Delta t}Q^{(i-1)} = {}^0\bar{K}\left({}^{t+\Delta t}\bar{K}^{(i-1)}\right)^{-1}\left({}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)}\right) \tag{21} $$

where, ${}^{t+\Delta t}Q^{(0)} = {}^{t+\Delta t}R - {}^tF$ and $i = 1, 2, 3, \ldots$. With the ELS obtained above and the initial stiffness matrix of which the inverse of it has already been computed, the following iterative compensation equation can be obtained which can replace the Newton-Raphson equation of Eqn 9:

$$ {}^0K\Delta U^{(i)} = {}^{t+\Delta t}Q^{(i-1)} \tag{22} $$

3.4. Computational Efficiency

The adaptive Newton-Raphson method proposed in this study computes ELS using the adaptive substructuring technique, and applies it to initial stiffness matrix to obtain compensated displacement. The efficiency of the proposed method depends on the ratio of the number of boundary DOFs, L, at nodal points connected to the elements with plastic deformation to the number of total DOFs, N.

In this section the computational efficiency of the proposed adaptive Newton-Raphson technique is compared with that of the conventional Newton-Raphson method, and a elasto-plastic analysis procedure is proposed to optimize the total efficiency of analysis. To compare computational efficiency the number of operations required for the repeated compensation in Eqns 17 and 22, especially the number of multiplication and division operations, are compared. It is assumed that banded stiffness matrix is used in the analysis. The solution processes of equilibrium equation are composed of decomposition of stiffness matrix, and reduction and back-substitution of a load vector. If N is the number of equations, $0.5N^3$ and $2N^2$ multiplication and division operations are required for decomposition of stiffness matrix and reduction and back-substitution of a load vector, respectively. If the band width of a stiffness matrix is $2M$, the number of operations is reduced to $0.5NM^2$ and $2NM$, respectively. Based on these factors, the comparison of operations required for repeated compensation is presented in Table 2. The conventional method requires formation of tangent stiffness matrix and its inversion in each analysis step, which results in $0.5NM^2$ multiplication and division operations; on the

Table 2. Comparison of number of operations between the conventional Newton-Raphson method and the adaptive Newton-Raphson method

Conventional Newton-Raphson method	Adaptive Newton-Raphson method		
Inverse of tangent stiffness (Eqn 9)	Flexibility matrix (Eqn 14)	Condensed stiffness (Eqn 15)	Equivalent load (Eqn 21)
$0.5NM^2$	$2NML$	$2.5L^3$	$0.5L^3$

M : Half band width of stiffness matrix, N : Total DOFs, L : Number of boundary DOFs

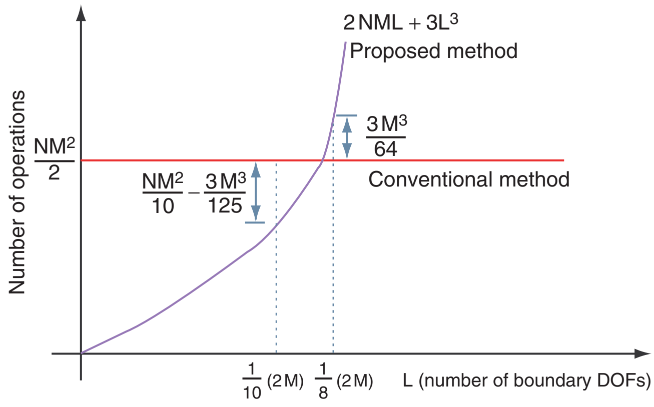


Figure 4. Number of operations of the conventional and the proposed method

other hand the proposed method requires operations of $2NML + 3L^3$. Therefore to compensate the out of balance load more effectively, the conventional method may be used when the former is smaller than the latter and vice versa. More simplified criterion is that when the number of boundary DOFs L is less than about 1/8 of the band width of the stiffness matrix $2M$, the proposed method will be more effective. The number of operations of the conventional plastic hinge method and the proposed method is compared graphically in Figure 4. The flow chart of the analysis procedure of a loading step and the proposed adaptive Newton-Raphson method is presented in Figure 5.

4. ANALYSIS OF AN EXAMPLE STRUCTURE

4.1. Model Structure and Applied Load

Figure 6 depicts a 3-story braced frame subjected to lateral load of R = 1,000 kN in each story. The member sizes are also shown in the figure. The force-displacement relationships of selected beams and braces are plotted in Figures 7(a) and 7(b), respectively. When the lateral loads are applied, the braces located in the first and second stories (members 10–13) undergo elastic-perfectly plastic behavior at tensile force of 2,500 kN and fail by buckling at compression force of 500 kN [Figure 7(b)]. When bending moment reaches 300 kN·m plastic hinges form at both ends of the second floor beam (member 3). The other members are assumed to behave elastically. The ELS for changing stiffness is obtained when the lateral load R reaches 500 kN and finally 1,000 kN.

4.2. ELS for Buckling of Braces

When the lateral load R reaches 500 kN as shown in Figure 8(a), the compressive forces in the braces 11 and 13 exceed 500 kN as shown in Figure 8(b) and the two braces buckle under compression. The bending moment diagram of the structure is shown in Figure 8(c). After buckling of braces the configuration of the structure changes as shown in Figure 9(a) and the stiffness of the structure also varies. In this case the boundary DOFs are horizontal and vertical

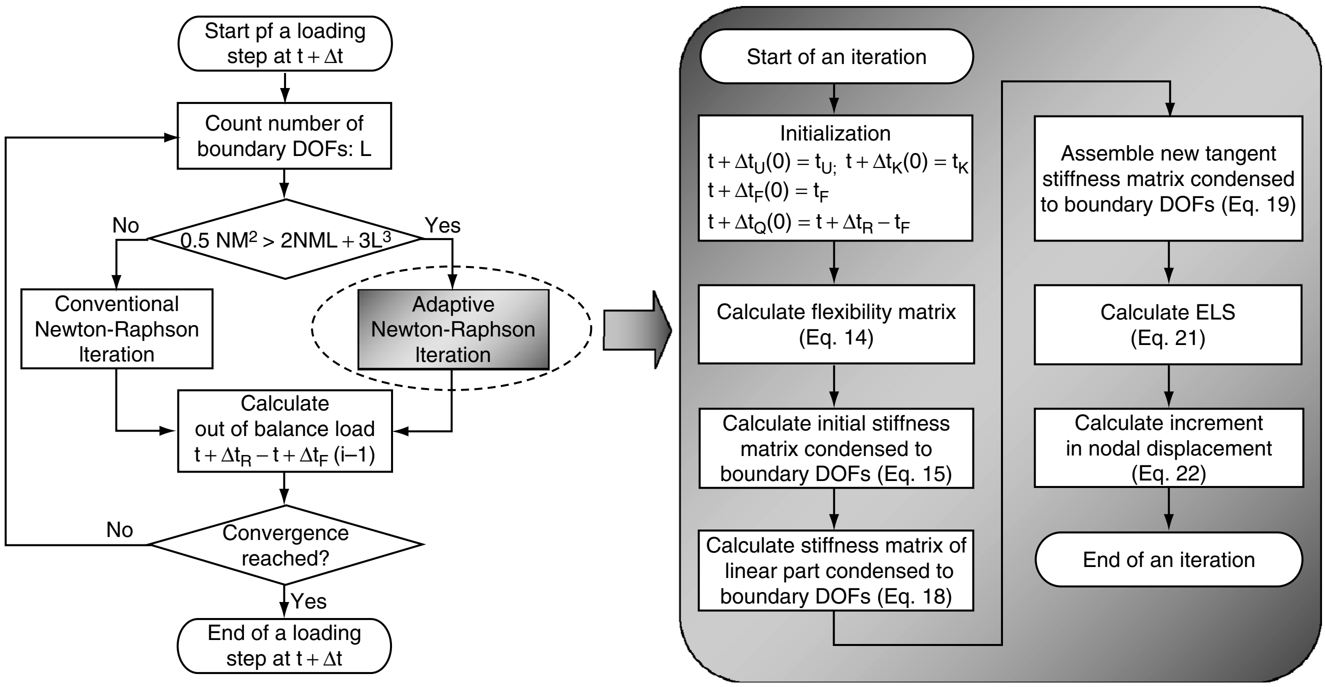


Figure 5. Flow chart of the adaptive Newton-Raphson iteration using ELS

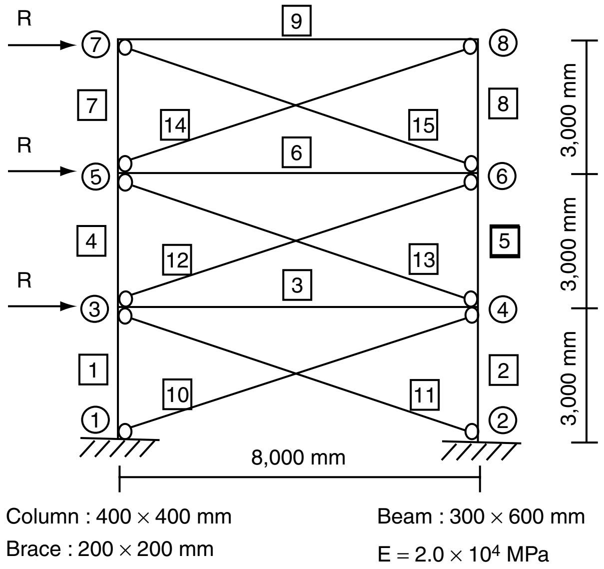


Figure 6. Analysis model structure

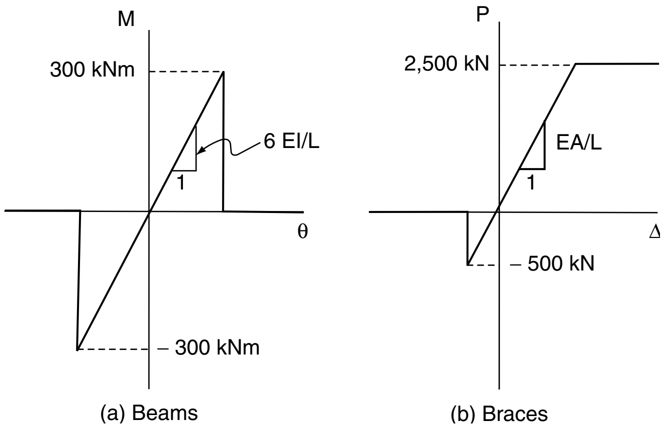


Figure 7. Load-displacement relationships of members

DOFs at joints 3, 4, and 5, and the ELS corresponding to the change in stiffness are presented in stage (1) of Table 3. Figures 9–10 compare the analysis results of both the model structure with buckled braces subjected to the initial lateral loads (P = 500 kN) and the initial structure subjected to the initial lateral loads and ELS. It can be observed that the final results are exactly identical. It also can be noticed that the application of the ELS generates member forces in the buckled braces, which has no physical meaning and can be neglected.

4.3. ELS for Plastic Hinges in the Beam

After buckling of the braces the forces resisted by the braces are transferred to other members; as shown in Figure 9(c) the bending moment at both ends of the beam number 3 exceeds 300 kN·m and plastic hinges form at both ends of the beam. Due to the formation of plastic hinges the rotational degrees of freedom at joints 3 and 4 become boundary DOFs, and now total of 8 boundary DOFs exist in the structure. The ELS computed in this stage are shown in stage (2) of Table 3. Figures 11–12 compare the analysis results of both the model structure with plastic hinges subjected to the initial lateral loads (P = 500 kN) and the initial structure subjected to the initial lateral loads and ELS. It can be observed that the final analysis results are exactly identical.

4.4. ELS for Yielding of Braces

When the lateral load R reaches 1,000 kN while the braces 11 and 13 were buckled and plastic hinges at both ends of the beam number 3 were formed [Figure 13(a)], the brace number 10 yields under tension. In this stage no boundary DOF is added and the ELS computed in

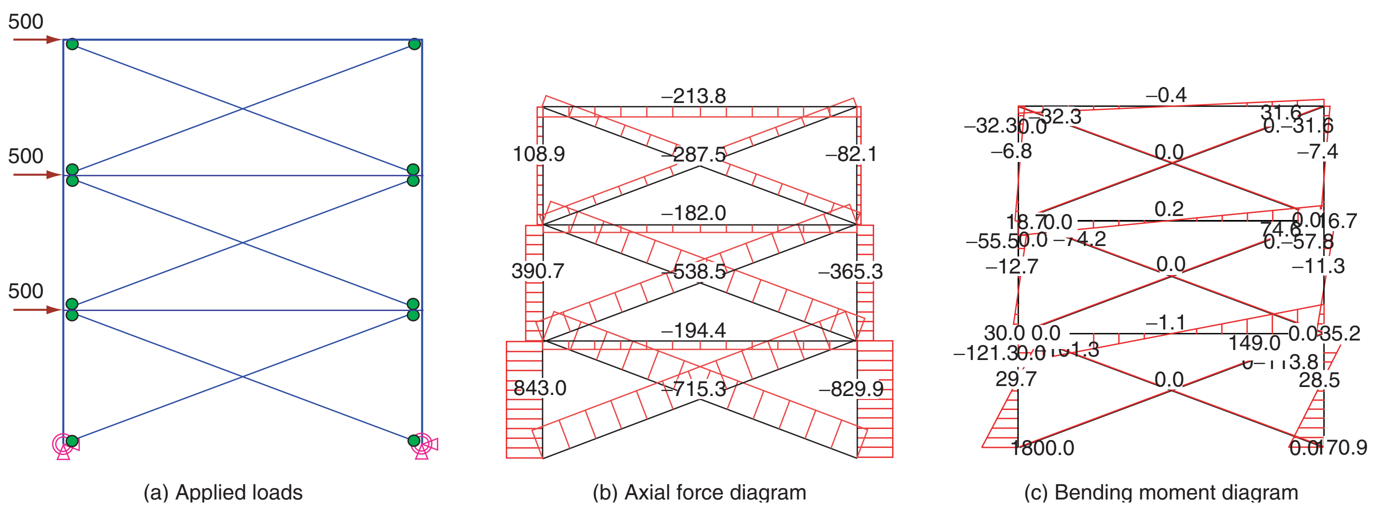
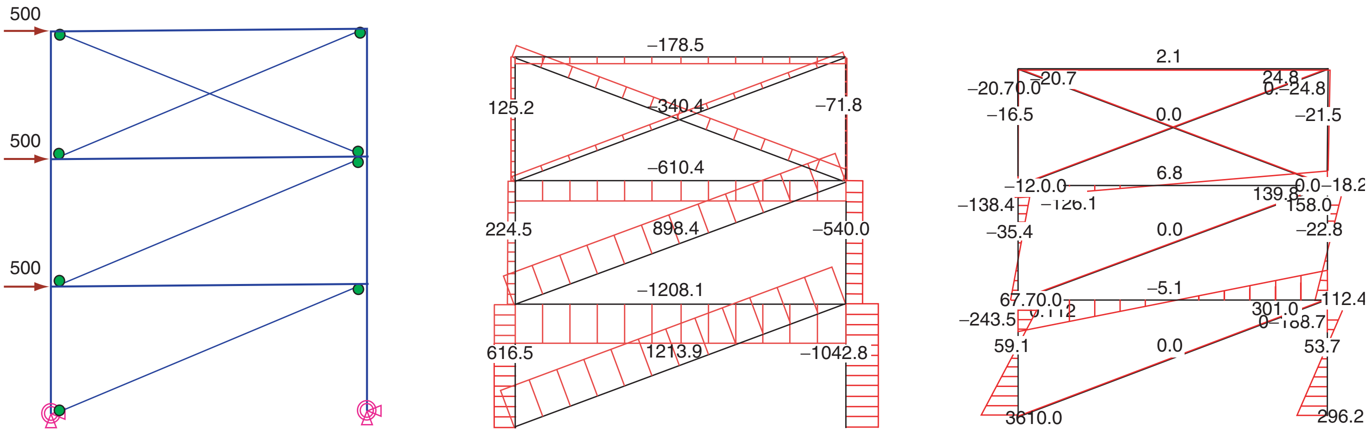


Figure 8. Model structure subjected to lateral load of 500 kN in each story (kN, m)

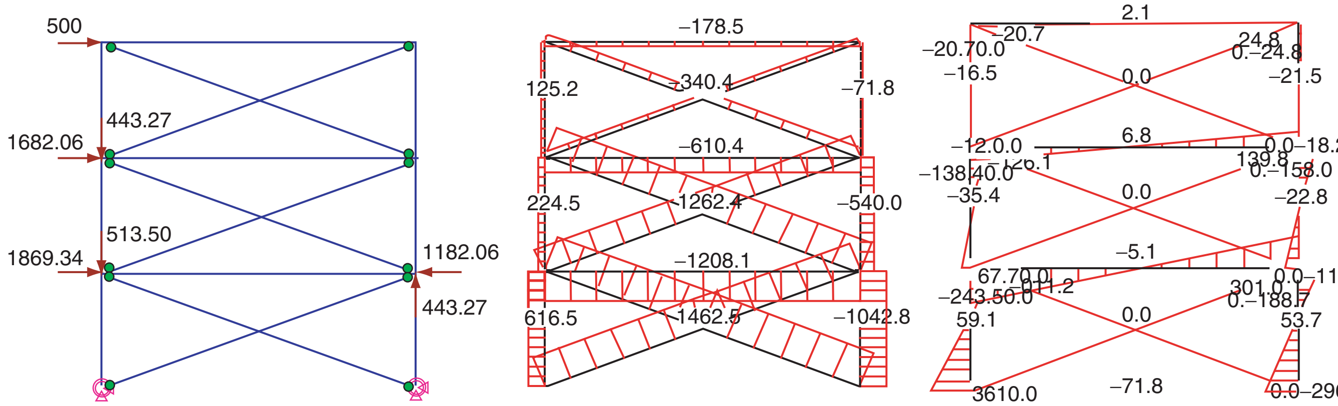


(a) Structure after stiffness change (b) Axial force diagram (c) Bending moment diagram

Figure 9. Conventional analysis after buckling of braces 11 and 13 (kN, m)

Table 3. ELS for stiffness change due to buckling and yielding of members (kN, m)

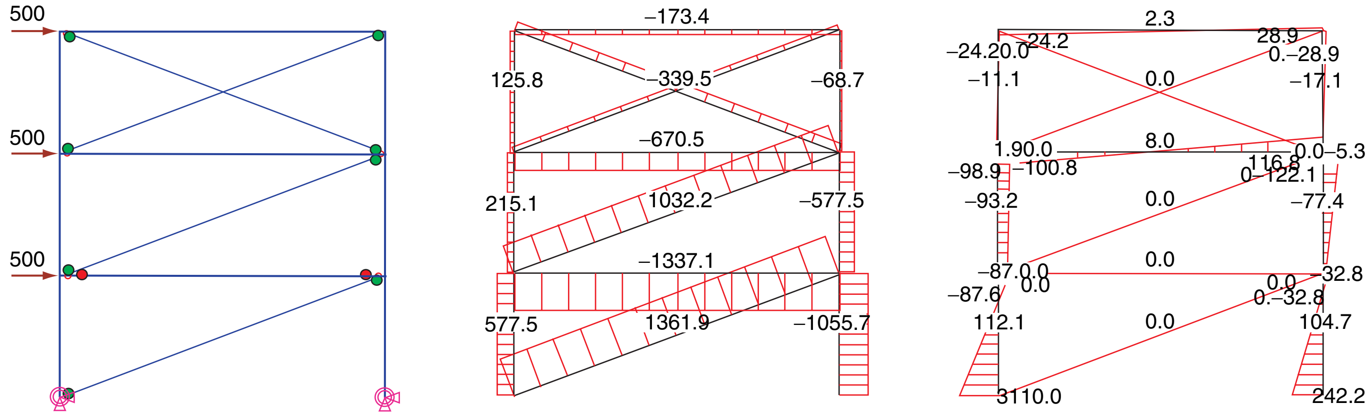
Stage status	DOF							
	3-X	3-Y	3-R	4-X	4-Y	4-R	5-X	5-Y
(1) Buckling of braces 11 and 13	1369.34	–513.50	–	–1182.06	443.27	–	1182.06	–443.27
(2) Stage (1) + Yielding of beam 3	1532.91	–725.60	–610.06	–1343.15	654.44	–596.0	1343.15	–503.68
(3) Stage (2) + Yielding of brace 10	3858.37	–1805.0	–1446.75	–2784.93	1402.47	–1418.17	3784.93	–1044.35



(a) ELS + applied loads (b) Axial force diagram (c) Bending moment diagram

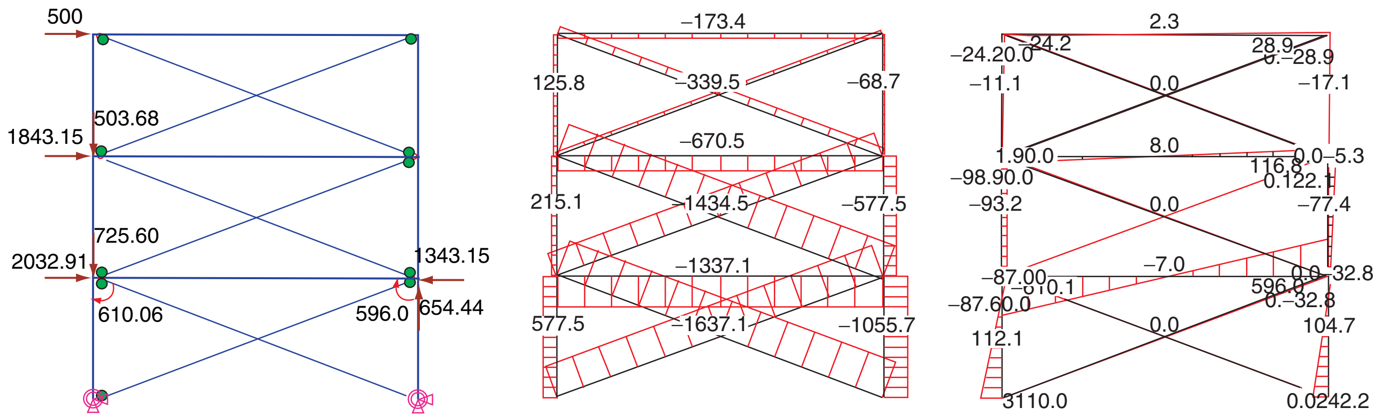
Figure 10. ELS based analysis for buckling of braces 11 and 13 (kN, m)

this stage are shown in stage (3) of Table 3. Figures 14–15 compare the analysis results of the model structure with a yielded brace subjected to the increased initial loads (R = 1,000 kN) and the initial structure subjected to the increased initial loads and ELS. The tensile yielding of the brace number 10 is considered by applying member yield force of 2,500 kN at joint 4 toward joint 1 as shown in Figure 14(a). The same technique is applied in the structure subjected to ELS as shown in Figure 15(a). It can be observed that, as before, the member forces obtained from both methods are exactly the same.



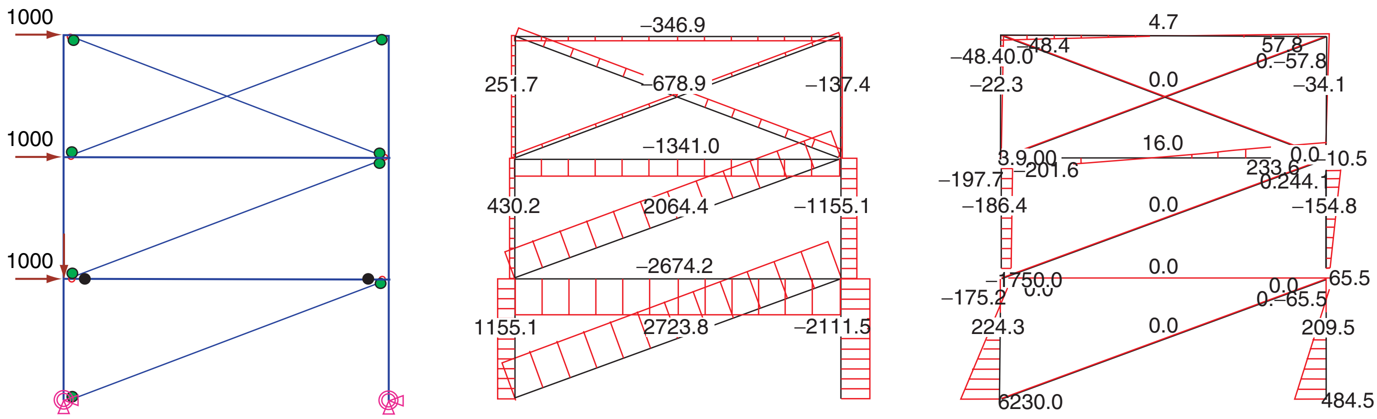
(a) Structure after stiffness change (b) Axial force diagram (c) Bending moment diagram

Figure 11. Conventional analysis after buckling of braces 11 and 13 and yielding of beam 3 (kN, m)



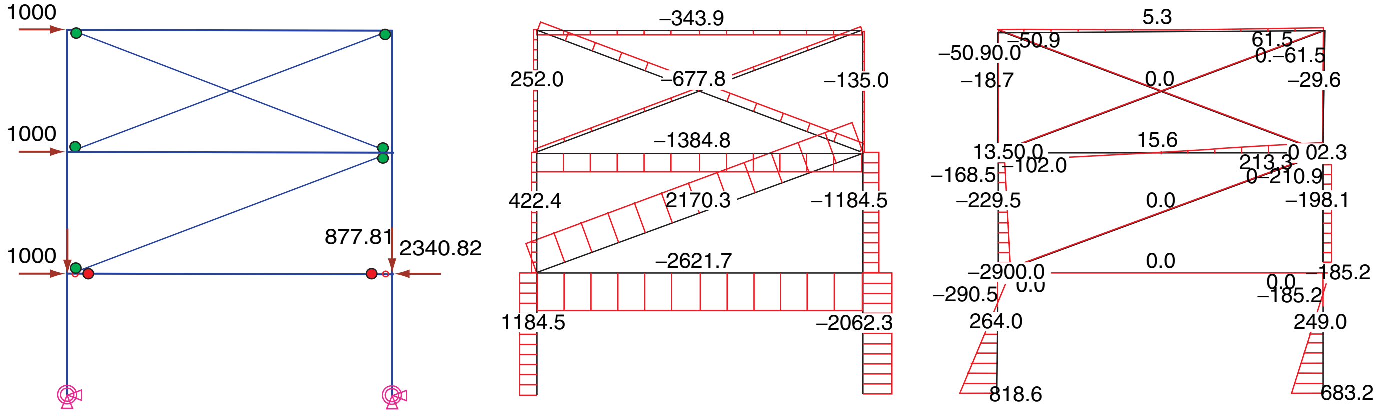
(a) ELS + applied loads (b) Axial force diagram (c) Bending moment diagram

Figure 12. ELS based analysis for buckling of braces 11 and 13 and plastic hinge formation in beam 3 (kN, m)



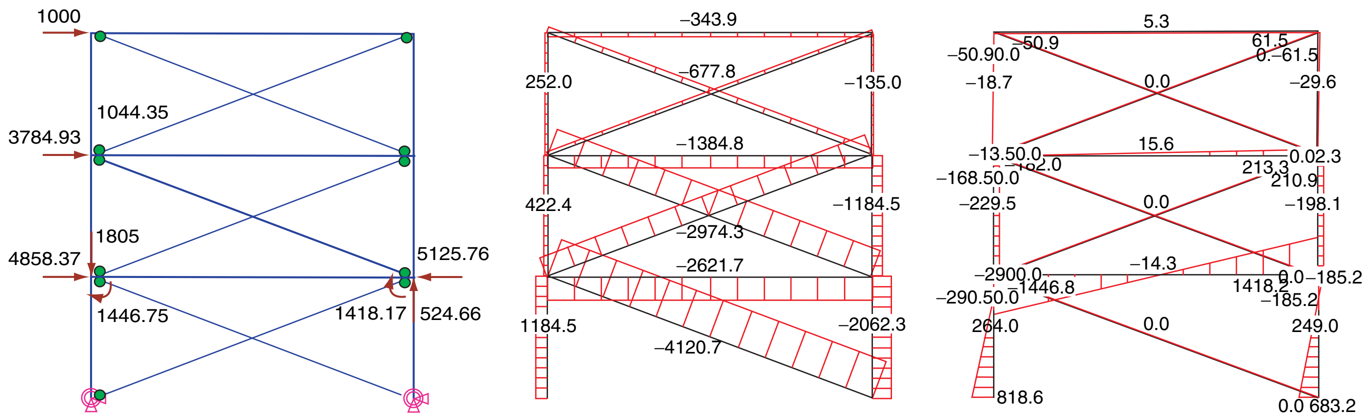
(a) Applied loads (b) Axial force diagram (c) Bending moment diagram

Figure 13. Model structure subjected to lateral load of 1,000 kN in each story (kN, m)



(a) Structure after stiffness change (b) Axial force diagram (c) Bending moment diagram

Figure 14. Conventional analysis after buckling of braces 11 and 13, flexural yielding of beam 3, and tensile yielding of brace 10 (kN, m)



(a) ELS + applied loads (b) Axial force diagram (c) Bending moment diagram

Figure 15. ELS based analysis for buckling of braces 11 and 13, yielding of beam 3, and tensile yielding of brace 10 (kN, m)

5. CONCLUSIONS

In this paper a new elasto-plastic analysis algorithm, the adaptive Newton-Raphson iteration method, for analysis of framed or braced structure is proposed. The presented algorithm utilizes the equivalent load for stiffness (ELS) corresponding to the stiffness change due to material nonlinear behavior and applies it to the initial global stiffness matrix to compute structural responses for nonlinear analysis. The analysis results of the model structure show that the linear analysis of a structure using the proposed algorithm produces identical responses as those by conventional plastic hinge analysis method.

The method can be very efficient as it calculates the inverse of the global stiffness matrix only once regardless of the number of loading steps. Also the sequential application of the load or the repeated compensation of the out of balance load is not required in the proposed algorithm. The efficiency of the proposed algorithm depends on the ratio of the boundary DOFs - the DOFs directly connected to the members of which the element stiffness are changed - to the total DOFs. When the number of boundary DOFs is less than about 1/8 of the band width of the stiffness matrix the proposed algorithm is more efficient than the conventional algorithm. Based on this DOF ratio (ratio of the boundary DOFs to the total DOFs), the combined use of the proposed algorithm and the conventional algorithm can enhance the overall computational efficiency of the analysis of structures with material nonlinearity.

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