Solutions for Homework #3

1. (1) The following strategic game models the situation.

the set of players:
{firm 1, firm 2, . . . , firm \( n \)}

players' sets of actions (or strategies), one for each player:
\([0, \infty)\) for each firm

players' payoff functions, one for each player:
\(\pi_i(q_1, \ldots, q_n) = P(Q)q_i - cq_i\) for firm \(i\)
or, more precisely,
\(\pi_i(q_1, \ldots, q_n) = (a - c)q_i - bQq_i\) if \(Q \leq a/b\)
\(-cq_i\) if \(Q > a/b\)

(2) Firm \(i\)'s payoff function is
\(\pi_i(q_1, \ldots, q_n) = (a - c)q_i - b(q_1 + \cdots + q_n)q_i\) if \(Q \leq a/b\)
\(-cq_i\) if \(Q > a/b\).

Firm \(i\)'s reaction function (or its best response function) is derived from
the first-order condition for maximizing \(\pi_i(q_1, \ldots, q_n)\),
and shows its best response for every possible combination (or collection)
of outputs that the other firms might choose.

Given any output \(Q_{-i} \in [0, (a - c)/b]\),
the first-order condition for maximizing \(\pi_i(q_1, \ldots, q_n)\) over \(q_i\) is
\((a - c) - bq_i - b(q_1 + \cdots + q_n) = 0\).

(\text{Note that, given } Q_{-i}, \text{ the maximand } \pi_i(q_1, \ldots, q_n) \text{ is strictly concave in } q_i, \text{ and thus the second-order condition for maximizing } \pi_i(q_1, \ldots, q_n) \text{ is satisfied.})

Thus firm \(i\)'s best response function is
\(b_i(Q_{-i}) \equiv b_i(Q_{-i}) = \begin{cases} (a - c)/2b - Q_{-i}/2 & \text{if } Q_{-i} \leq (a - c)/b \\ 0 & \text{if } Q_{-i} > (a - c)/b, \end{cases}\)

where \(q_{-i}\) stands for the list of the outputs of all the firms except firm \(i\).

(You may write this as
\(q_i = \begin{cases} (a - c)/2b - Q_{-i}/2 & \text{if } Q_{-i} \leq (a - c)/b \\ 0 & \text{if } Q_{-i} > (a - c)/b, \end{cases}\)
We obtain
\[ db_i(Q_{-i})/dQ_{-i} = -1/2 < 0 \quad \text{for all } Q_{-i} \in [0, (a - c)/b]. \]

Let \( q_1^m \) denote firm 1's profit-maximizing output when all the other firms produce zero outputs. That is, \( q_1^m \) is firm 1's best response to
\[ Q_{-1} \equiv q_2 + \cdots + q_n = 0. \]
Then we obtain \( q_1^m = (a - c)/2b. \)

We show that any output of firm 1 which is larger than its output \( q_1^m \) is strictly dominated by \( q_1^m \), and also that no output of firm 1 which is smaller than or equal to \( q_1^m \) is strictly dominated.

\[ \ast \quad \text{Let } q'_1 \text{ be firm 1's output with } q'_1 > q_1^m. \]
Then given any output \( Q_{-1} \in R^+ \), firm 1's profit from choosing \( q'_1 \) is
\[ \pi'_1 = P(q'_1 + Q_{-1})q'_1 - c q'_1, \]
and its profit from choosing \( q_1^m \) is
\[ \pi^m_1 = P(q_1^m + Q_{-1})q_1^m - c q_1^m. \]
It is straightforward to see that
\[ \pi'_1 < \pi^m_1 \quad \text{for any output } Q_{-1} \in R^+. \]

\[ \ast \quad \text{Let } q''_1 \text{ be firm 1's output with } 0 \leq q''_1 \leq q_1^m. \]
Then \( q''_1 \) is firm 1's best response to some output \( Q_{-1} \in [0, (a - c)/b]. \).
That is, given some output \( Q_{-1} \in [0, (a - c)/b] \), firm 1's profit with \( q''_1 \) exceeds its profit with any other output.
This implies that \( q''_1 \) is not strictly dominated (nor is it weakly dominated).

To find the Nash equilibrium, we use the firms' best response functions. Specifically, we find it by solving the system of \( n \) simultaneous equations which come from the firms' best response functions.

The conditions for \( (q_1^*, \ldots, q_n^*) \) to be the Nash equilibrium are
\[ q_1^* = b_1(q_{-1}^*) \]
\[ \vdots \]
\[ q_n^* = b_n(q_{-n}^*). \]
More specifically, in the Nash equilibrium in which the firms' outputs are positive, we have

\[ q_1^* = \frac{(a - c) - (q_2^* + q_3^* + \cdots + q_n^*)}{2} \]
\[ q_2^* = \frac{(a - c) - (q_1^* + q_3^* + \cdots + q_n^*)}{2} \]
\[ \vdots \]
\[ q_n^* = \frac{(a - c) - (q_1^* + q_2^* + \cdots + q_{n-1}^*)}{2}. \]

Now let us solve these equations to find the Nash equilibrium. Subtracting the second equation from the first, we obtain \( q_1^* = q_2^* \).

Similarly, subtracting the third equation from the second, we conclude that \( q_2^* = q_3^* \).

Continuing with all pairs of equations, we deduce that \( q_1^* = q_2^* = \cdots = q_n^* \).

Let each firm's output be \( q^* \).

Then \( q^* = \frac{(a - c) - b(n + 1)}{b(n + 1)}. \)

In summary, the game has a unique Nash equilibrium, in which each firm produces \( \frac{(a - c)}{b(n + 1)} \):

\( (q_1^*, \ldots, q_n^*) = ((a - c)/b(n + 1), \ldots, (a - c)/b(n + 1)). \)

(7) Yes, there are combinations of the firms' outputs that yield higher profits for all the firms than do the Nash equilibrium outputs. For example, the following combination of outputs is one of them:

\( (q_1, \ldots, q_n) = ((a - c)/2bn, \ldots, (a - c)/2bn), \)

in which the optimal monopoly output is split equally among the \( n \) firms.

(8) The price at the Nash equilibrium is

\( P(Q^*) = \frac{(a + nc)}{(n + 1)}. \)

We have

\( \frac{\partial P(Q^*)}{\partial n} = (c - a)/(n + 1)^2 < 0. \)

This means that the price decreases as \( n \) increases.

Using a technique known as L'Hôpital's rule, we obtain

\( \lim_{n \to \infty} P(Q^*) = c. \)

This means that the price approaches \( c \) as \( n \) increases without bound.
2. (1) The following strategic game models the situation.

- the set of players: 
  \{player 1, player 2, \ldots, player n\}, where \(n \geq 2\)
- the players' sets of actions (or strategies), one for each player:
  \([0, \infty)\) for each player
- the players' payoff functions, one for each player:
  \[\pi_i(b_1, \ldots, b_n) = v_i - b_i\]
  if \(b_i > \overline{b}_{-i}\), or
  if \(b_i = \overline{b}_{-i}\) and the number of every other player who bids \(\overline{b}_{-i}\) is greater than \(i\)
  \[\pi_i(b_1, \ldots, b_n) = 0\] otherwise

(2) We show that, for any bid \(b_i > v_i\), player \(i\)'s bid \(v_i\) is at least as good as \(b_i\), no matter what the other players bid, and is better than \(b_i\) for some bids of the other players.

Let \(b'_i\) be player \(i\)'s bid greater than \(v_i\).

We compute player \(i\)'s payoffs to her bid \(v_i\), and those to her bid \(b'_i\), as a function of \(\overline{b}_{-i}\).

<table>
<thead>
<tr>
<th>(i)'s bid</th>
<th>(\overline{b}_{-i} &lt; v_i)</th>
<th>(\overline{b}_{-i} = v_i)</th>
<th>(v_i &lt; \overline{b}_{-i} &lt; b'_i)</th>
<th>(\overline{b}_{-i} = b'_i)</th>
<th>(\overline{b}_{-i} &gt; b'_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b'_i)</td>
<td>(v_i - b'_i)</td>
<td>(v_i - b'_i)</td>
<td>(v_i - b'_i)</td>
<td>((v_i - b'_i)) or 0</td>
<td>0</td>
</tr>
<tr>
<td>(v_i)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that \(v_i - b'_i\) is negative.

Comparing player \(i\)'s payoffs to her bid \(v_i\) and her payoffs to her bid \(b'_i\), we obtain the following:

\[\pi_i(b_1, \ldots, b_{i-1}, v_i, b_{i+1}, \ldots, b_n) \geq \pi_i(b_1, \ldots, b_{i-1}, b'_i, b_{i+1}, \ldots, b_n)\]

for all \(\overline{b}_{-i}\) in \(R^+\),

and

\[\pi_i(b_1, \ldots, b_{i-1}, v_i, b_{i+1}, \ldots, b_n) > \pi_i(b_1, \ldots, b_{i-1}, b'_i, b_{i+1}, \ldots, b_n)\]

for, for example, \(0 \leq \overline{b}_{-i} < b'_i\).
This says that, for all values of $b_{-i}$, player $i$'s payoffs to her bid $v_i$ are at least as large as her payoffs to $b_i'$ and, for $0 \leq b_{-i} < b_i'$, her payoffs to $v_i$ exceed her payoffs to $b_i'$.

Thus player $i$'s bid $v_i$ weakly dominates her bid greater than her valuation.

(3) Let $b_i''$ be player $i$'s positive bid less than her valuation $v_i$.
We compute player $i$'s payoffs to her bid $v_i$, and those to her bid $b_i''$.

<table>
<thead>
<tr>
<th>$i$'s bid</th>
<th>$b_{-i} &lt; b_i''$</th>
<th>$b_{-i} = b_i''$</th>
<th>$b_i'' &lt; b_{-i} &lt; v_i$</th>
<th>$b_{-i} = v_i$</th>
<th>$b_{-i} &gt; v_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_i$</td>
<td>0</td>
<td>0</td>
<td>$0$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$b_i''$</td>
<td>$v_i - b_i''$</td>
<td>$(v_i - b_i'')$ or 0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that $v_i - b_i''$ is positive.

Comparing player $i$'s payoffs to her bid $v_i$ and her payoffs to her bid $b_i''$, we obtain the following:

$$\pi_i(b_1, \ldots, b_{i-1}, v_i, b_{i+1}, \ldots, b_n) \leq \pi_i(b_1, \ldots, b_{i-1}, b_i'', b_{i+1}, \ldots, b_n)$$
for all $b_{-i} \in R_+$,

and

$$\pi_i(b_1, \ldots, b_{i-1}, v_i, b_{i+1}, \ldots, b_n) < \pi_i(b_1, \ldots, b_{i-1}, b_i'', b_{i+1}, \ldots, b_n)$$
for, for example, $0 \leq b_{-i} < b_i''$.

This says that, for all values of $b_{-i}$, player $i$'s payoffs to her positive bid $b_i''$ are at least as large as her payoffs to $v_i$ and, for $0 \leq b_{-i} < b_i''$, her payoffs to $b_i''$ exceed her payoffs to $v_i$.

Thus player $i$'s positive bid $b_i''$ weakly dominates her bid of $v_i$. 