Contests with Alternative Public-Good Prizes

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DOI: 10.1111/jpet.12196

Abstract

I study contests in which a society of players compete, by expending irreversible effort, over which one of alternative prizes to have awarded to them by the decision maker. The prizes are public goods and/or public bads for the players. The players choose their effort levels simultaneously and independently. I define each player's valuation spread as the difference between his valuations for the two public-good/public-bad prizes. I establish that the players' equilibrium effort levels depend solely on their valuation spreads, and that the players never expend positive effort for both prizes in equilibrium. Further, I establish that, in equilibrium, only players with the widest positive valuation spread and players with the widest negative valuation spread expend positive effort. Finally, I establish that the equilibrium effort level expended for each prize and the equilibrium total effort level are determined only by the widest positive valuation spread and the widest negative valuation spread.

JEL classification: D72, H41, C72

Keywords: Contest; Rent Seeking; Public-good prize; Public-bad prize; Valuation spread; Stake; The free-rider problem

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1. Introduction

Consider a situation in which there are two candidates who compete to be the president of a country and rent seekers make contributions to the candidates' election campaign. The contributions are nonrefundable and are used to finance the candidates' election campaigns. The winning candidate is selected according to a rule which is based on the voluntary contributions the rent seekers make. The rent seekers all are affected by the election result. Some of them get benefits from the elected president – they are granted government budgets, favors, monopoly rights, etc. Some of them get damage from the elected president. This means that each of the presidential candidates can be considered as a public-good or public-bad prize for each rent seeker.

Contests involving public-good and/or public-bad prizes, more specifically those like the motivating example above, are easily observed in the real world. An example is contests in which people compete over which one of alternative public goods to have provided for them by their local government. Another example is contests in which lobbyists or firms compete over which one of governmental trade policies to have adopted.

This paper studies such contests. In particular, it focuses on examining what factors determine contestants' effort expended for alternative public-good/bad prizes, how much effort each contestant expends, and how severe the free-rider problem is.

This paper formally considers the following simultaneous-move game. There are alternative public-good/bad prizes which will be awarded to a society of players. The players compete, by expending irreversible effort for the prizes, over which prize to have awarded or selected by the decision maker. At the beginning of the game, each player's valuations for the prizes are publicly known. The players are risk-neutral, and are assumed to have different valuations for the prizes.

Define each player's valuation spread as the difference between his valuations for the two public-good/bad prizes. This paper first establishes that only the players' valuation spreads matter in equilibrium: The players' equilibrium effort levels do not depend separately on their...
valuations for one prize and their valuations for the other prize, but only on their valuation spreads between the prizes. Next, this paper establishes that the players never expend positive effort for both prizes in equilibrium. To put it more interestingly, the players take stands on one side or the other — that is, no straddling occurs at all. Next, this paper establishes that, in equilibrium, only players with the widest positive valuation spread and players with the widest negative valuation spread — that is, only players with the strongest preference for one prize over the other — expend positive effort; a player whose valuation spread, positive or negative, is narrower than somebody else's expends zero effort for both prizes. Clearly, this suggests that the free-rider problem is prevalent and severe. Finally, this paper establishes that the equilibrium effort level expended for each prize and the equilibrium total effort level are determined only by the widest positive valuation spread and the widest negative valuation spread. All this implies that each player's equilibrium pair of effort levels is independent of the number of players, the players' valuations for each prize, the sum of the valuations, and the distribution of the valuations in the society, unless changes in these come with a change in the widest positive or negative valuation spread.

This paper is related to Katz, Nitzan, and Rosenberg (1990), Ursprung (1990), Baik (1993), Riaz, Shogren, and Johnson (1995), Baik and Shogren (1998), Baik, Kim, and Na (2001), Baik (2008), Epstein and Mealem (2009), Lee (2012), Kolmar and Rommeswinkel (2013), Chowdhury, Lee, and Sheremeta (2013), Topolyan (2014), and Barbieri, Malueg, and Topolyan (2014). These papers study contests in which *groups* of players compete to win a *single* prize and the prize is a public good only *within each group*. In the papers, the number of groups and their sizes are exogenous. Baik (1993), Baik and Shogren (1998), Baik et al. (2001), and Baik (2008) establish that the free-rider problem is of the severest form in equilibrium — that is, in each group, all the players except highest-valuation players free ride. On the other hand, others such as Lee (2012) and Topolyan (2014) establish that the free-rider problem does not exist in equilibrium. Striking differences between the current paper and the previous papers are threefold. First, most importantly, there are *alternative* public-good/bad prizes in the current
paper. Second, the prizes are public goods and/or bads for all the players, and are not group-specific, in the current paper. Third, in the current paper, groups are not assumed to exist at the outset; however, two sets of players may be endogenously identified depending on the players' valuation spreads: the set of players with zero or a positive valuation spread and the set of players with a negative valuation spread.

The paper proceeds as follows. Section 2 develops the model, and sets up the simultaneous-move game. In Section 3, I solve for, and characterize, the pure-strategy Nash equilibria of the game. Finally, Section 4 presents conclusions.

2. The model

I consider a contest in which one, and only one, of alternative prizes will be awarded to a society of $n$ players, 1 through $n$, and the players "compete" over which prize to have awarded or selected by the decision maker, where $n \geq 2$. The alternative prizes are called prizes $A$ and $B$, respectively, and are public goods or public bads for the players — specifically, each prize may be a public good for some players and a public bad for the others. The players are risk-neutral, and expend effort or make contributions for the prizes. Each player cannot recover his effort expended whether prize $A$ or prize $B$ is selected. The players' valuations for the prizes are publicly known. Let $v_i^J$ represent player $i$'s valuation for prize $J$, for $J = A, B$, where $v_i^J$ is negative or zero or positive. Let me call $(v_i^A - v_i^B)$ player $i$'s valuation spread between prizes $A$ and $B$.

**Assumption 1.** I assume, without loss of generality, that $(v_1^A - v_1^B) \geq \ldots \geq (v_n^A - v_n^B)$.

I assume that $(v_1^A - v_1^B)(v_1^B - v_1^B) < 0$.

The second part of Assumption 1 is assumed for concise exposition. It eliminates from consideration trivial cases, such as a case where $(v_1^A - v_1^B) \geq 0$ and a case where $(v_1^A - v_1^B) \leq 0$. 
Note that \((v^A_n - v^B_n) > 0\) would imply that all the players have strong preference for prize A over prize B, while \((v^A_1 - v^B_1) < 0\) would imply the opposite.

Let \(x_i\) denote the effort level expended for prize A by player \(i\), and let \(X\) denote the effort level expended for prize A by all the players. Let \(y_i\) denote the effort level expended for prize B by player \(i\), and let \(Y\) denote the effort level expended for prize B by all the players. Then, in terms of the symbols, I have \(X \equiv \sum_{i=1}^n x_i\) and \(Y \equiv \sum_{i=1}^n y_i\). Each player's effort level for prize \(J\), for \(J = A, B\), is nonnegative, and is measured in units commensurate with the prizes. Let \(p^J\) represent the probability that prize \(J\) is selected, where \(0 \leq p^J \leq 1\) and \(p^A + p^B = 1\). The selection probability of prize \(J\) — that is, the probability of prize \(J\) being selected — depends on the players' effort levels for both prizes. I assume that the selection probability function for prize \(J\) is \(p^J = p^J(X, Y)\), where the function \(p^J\) has the properties specified in Assumption 2 below.

**Assumption 2.** (a) I assume that \(\partial p^A / \partial X \geq 0\), \(\partial p^A / \partial Y \leq 0\), \(\partial^2 p^A / \partial X^2 \leq 0\), and \(\partial^2 p^A / \partial Y^2 \geq 0\). I assume that \(\partial p^A / \partial X > 0\) and \(\partial^2 p^A / \partial X^2 < 0\) when \(Y > 0\), and that \(\partial p^A / \partial Y < 0\) and \(\partial^2 p^A / \partial Y^2 > 0\) when \(X > 0\). (b) I assume that \(\partial p^B / \partial X \leq 0\), \(\partial p^B / \partial Y \geq 0\), \(\partial^2 p^B / \partial X^2 \geq 0\), and \(\partial^2 p^B / \partial Y^2 \leq 0\). I assume that \(\partial p^B / \partial X < 0\) and \(\partial^2 p^B / \partial X^2 > 0\) when \(Y > 0\), and that \(\partial p^B / \partial Y > 0\) and \(\partial^2 p^B / \partial Y^2 < 0\) when \(X > 0\).

In Assumption 2, I assume that, given the effort level expended for the rival prize, each prize's selection probability is increasing in the effort level for itself at a decreasing rate. I also assume that, given the effort level expended for itself, each prize's selection probability is decreasing in the effort level for the rival prize at a decreasing rate.

Let \(\pi_i\) denote the expected payoff for player \(i\). Then the payoff function for player \(i\) is

\[
\pi_i = v^A_i p^A(X, Y) + v^B_i [1 - p^A(X, Y)] - x_i - y_i.
\]

\[
= v^B_i + (v^A_i - v^B_i) p^A(X, Y) - x_i - y_i. \quad (1)
\]
I assume that all the players choose their effort levels simultaneously and independently — that is, when a player chooses his effort levels for prizes $A$ and $B$, respectively, he does not know the other players' effort levels. I also assume that all of the above is common knowledge among the players. Finally, I employ Nash equilibrium as the solution concept.

3. The players' valuation spreads do matter!

In this section, I obtain the pure-strategy Nash equilibrium or equilibria of the game, and establish, among other things, that the players' equilibrium effort levels depend solely on their valuation spreads. I establish also that, in equilibrium, the player or players with the widest positive valuation spread and the player or players with the widest negative valuation spread expend positive effort, while the rest expend zero effort and free ride.

I begin by showing that, in equilibrium, each player expends zero effort for at least one of the two prizes. Let $\mathcal{P}$ represent the set of players with zero or a positive valuation spread, and let $\mathcal{N}$ represent the set of players with a negative valuation spread. Then, in terms of the symbols, I have $v_i^A - v_i^B \geq 0$ for $i \in \mathcal{P}$ and $v_j^A - v_j^B < 0$ for $j \in \mathcal{N}$. Loosely speaking, $\mathcal{P}$ is the set of players who have preference for prize $A$ over prize $B$, while $\mathcal{N}$ is the set of players who have preference for prize $B$ over prize $A$. By Assumption 1, neither set is empty, so that players 1 through $k$ belong to $\mathcal{P}$ and players $k+1$ through $n$ belong to $\mathcal{N}$, where $k$ equals the size of the set $\mathcal{P}$.

**Lemma 1.** In equilibrium, (a) player $i$, for $i \in \mathcal{P}$, expends zero effort for prize $B$, and (b) player $j$, for $j \in \mathcal{N}$, expends zero effort for prize $A$.

The proof of Lemma 1 is provided in Appendix A. Part (a) can be explained as follows. Each player in $\mathcal{P}$ has preference for prize $A$ over prize $B$, and thus tries his best for prize $A$ to be selected — or, for prize $B$ not to be selected. Since effort is nonnegative, the best he can do against prize $B$ is to expend zero effort for it. Another way to explain it is that the players in $\mathcal{P}$
can enjoy at least prize $B$ regardless of their effort, and thus expend zero effort for prize $B$. Part $(b)$ can be explained similarly.

Based on Lemma 1, in order to obtain the pure-strategy Nash equilibria of the game, I will henceforth restrict attention only to the strategy profiles at which players $1$ through $k$ expend zero effort for prize $B$ and players $k+1$ through $n$ expend zero effort for prize $A$.

As an informative preliminary step to obtain the pure-strategy Nash equilibria of the game, I first obtain its $\mathbb{P}$-specific quasi-equilibria and its $\mathbb{N}$-specific quasi-equilibria.

### 3.1. $\mathbb{P}$-specific quasi-equilibria

Consider the players in $\mathbb{P}$, players $1$ through $k$, each of whom has zero or a positive valuation spread. For $i \in \mathbb{P}$, player $i$'s valuation spread $(v_i^A - v_i^B)$ is also called his stake. Because I restrict attention only to the strategy profiles at which $y_i = 0$ for $i \in \mathbb{P}$ and $x_j = 0$ for $j \in \mathbb{N}$, $X$ is now the effort level expended for prize $A$ only by the players in $\mathbb{P}$, and $Y$ is now the effort level expended for prize $B$ only by the players in $\mathbb{N}$. For $i \in \mathbb{P}$, let $x_i^b$ represent player $i$'s best response to the other players' effort levels (including those expended for prize $B$ by the players in $\mathbb{N}$). Then, because $x_i^b$ is player $i$'s effort level expended for prize $A$ which maximizes his expected payoff

$$
\pi_i = v_i^B + (v_i^A - v_i^B) p^A(X, Y) - x_i
$$

subject to $x_i \geq 0$, it satisfies the following first-order condition:

$$(v_i^A - v_i^B) (\partial p^A / \partial x_i) - 1 = (v_i^A - v_i^B) (\partial p^A / \partial X) - 1 = 0 \quad \text{for } x_i^b > 0 \quad (2)$$

or

$$(v_i^A - v_i^B) (\partial p^A / \partial x_i) - 1 = (v_i^A - v_i^B) (\partial p^A / \partial X) - 1 \leq 0 \quad \text{for } x_i^b = 0. \quad (3)$$

In the case where $x_i^b > 0$, player $i$'s marginal gross payoff, $(v_i^A - v_i^B)(\partial p^A / \partial x_i)$, is equal to his marginal cost, $1$, at the positive effort level, $x_i^b$. In the case where $x_i^b = 0$, his marginal gross payoff does not exceed his marginal cost at the zero effort levels. Because player $i$'s marginal
gross payoff, \((v_i^A - v_i^B)(\partial p^A / \partial x_i)\), decreases in his effort level, \(x_i\), due to Assumption 2, his payoff function is strictly concave in his effort level. This implies that the second-order condition for maximizing \(\pi_i\) is satisfied, and also that \(x_i^*\) is unique.

Define a \(\mathbb{P}\)-specific quasi-equilibrium, denoted by \((x_1^*, \ldots, x_k^*)\), as a \(k\)-tuple vector of effort levels expended for prize \(A\), one for each player in \(\mathbb{P}\), at which \(x_i^*\), for \(i \in \mathbb{P}\), is the best response to the other players' effort levels (including those expended for prize \(B\) by the players in \(\mathbb{N}\)).

To obtain \(\mathbb{P}\)-specific quasi-equilibria, I begin by defining the optimal effort level for player \(i\) given the effort level \(Y\) of the players in \(\mathbb{N}\), where \(i \in \mathbb{P}\).

**Definition 1.** For \(i \in \mathbb{P}\), let \(X^b(Y; i)\) denote the optimal effort level for player \(i\) given the effort level \(Y\) of the players in \(\mathbb{N}\) (or, for short, player \(i\)'s optimal effort given \(Y\)). I define \(X^b(Y; i)\) as the effort level that maximizes

\[
v_i^B + (v_i^A - v_i^B) p^A(X, Y) - X
\]

subject to \(X \geq 0\).

Player \(i\)'s optimal effort given \(Y\) can be considered as the best response of the set \(\mathbb{P}\) of players as a whole to the effort level \(Y\) of the players in \(\mathbb{N}\) when \(v_i^A\) and \(v_i^B\) are taken as \(\mathbb{P}\)'s valuations for prizes \(A\) and \(B\), respectively. Simply speaking, it represents \(\mathbb{P}\)'s best response to \(Y\) which is computed using player \(i\)'s valuations for the prizes. Accordingly, \(X^b(Y; i)\) satisfies the following first-order condition:

\[
(v_i^A - v_i^B)(\partial p^A / \partial X) - 1 = 0 \quad \text{for } X^b(Y; i) > 0
\]

or

\[
(v_i^A - v_i^B)(\partial p^A / \partial X) - 1 \leq 0 \quad \text{for } X^b(Y; i) = 0.
\]

Because, under Assumption 2, \((v_i^A - v_i^B)(\partial p^A / \partial X)\) in (5) and (6) decreases in \(X\), (4) is strictly concave in \(X\). This implies that the second-order condition for maximizing (4) is satisfied, and also that \(X^b(Y; i)\) is unique.
Next, in Lemma 2, I compare $X^b(Y; h)$ with $X^b(Y; t)$, for $h, t \in \mathbb{P}$ with $h \neq t$. In other words, I compare $\mathbb{P}$'s best responses to $Y$, each computed using a different player's valuations for prizes $A$ and $B$.

**Lemma 2.** Given the effort level $Y$, $\mathbb{P}$'s best response computed using a wider positive valuation spread (or a higher stake) is greater than or equal to that computed using a narrower positive valuation spread (or a lower stake): $X^b(Y; s-1) \geq X^b(Y; s)$ for $s = 2, \ldots, k$.

The proof of Lemma 2 is provided in Appendix B. Lemma 2 says that, given the effort level $Y$ of the players in $\mathbb{N}$, $X^b(Y; 1)$ is the largest, $X^b(Y; 2)$ is second largest, $X^b(Y; 3)$ third, and so on, with $X^b(Y; k)$ being the smallest. Loosely speaking, Lemma 2 holds because, given the effort level $Y$, $\mathbb{P}$'s gross marginal payoff, $(v_h^d - v_h^P)(\partial p^d/\partial X)$, computed using a wider positive valuation spread (or a higher stake) is greater than its gross marginal payoff, $(v_i^d - v_i^P)(\partial p^d/\partial X)$, computed using a narrower positive valuation spread (or a lower stake) at any effort level, while $\mathbb{P}$'s marginal cost of increasing effort is constant.

Next, in Lemma 3, I demonstrate that the effort level of the players in $\mathbb{P}$, $X^* \equiv \sum_{i=1}^{k} x_i^*$, at a $\mathbb{P}$-specific quasi-equilibrium given the effort level $Y$ must be equal to player 1's optimal effort given $Y$: $X^* = X^b(Y; 1)$. This result is useful in obtaining $\mathbb{P}$-specific quasi-equilibria given $Y$ in Lemma 4.

**Lemma 3.** The effort level $X^*$ of the players in $\mathbb{P}$ at a $\mathbb{P}$-specific quasi-equilibrium given $Y$ is equal to $X^b(Y; 1)$.

The proof of Lemma 3 is provided in Appendix C. Lemma 3 says that, given an effort level of the players in $\mathbb{N}$, the effort level of the players in $\mathbb{P}$ at a $\mathbb{P}$-specific quasi-equilibrium is exactly the same as the optimal effort level for player 1, the player with the widest positive
valuation spread (or the highest stake). Consequently, the effort level of the players in \( P \) at a \( P \)-specific quasi-equilibrium given \( Y \) depends only on the widest of the valuation spreads (or the highest of the stakes) that the players in \( P \) have.

Finally, in Lemma 4, I obtain \( P \)-specific quasi-equilibria given \( Y \). Recall that a \( P \)-specific quasi-equilibrium given \( Y \) is defined as a \( k \)-tuple vector, \( (x_1^*, \ldots, x_k^*) \), of effort levels expended for prize \( A \), one for each player in \( P \). At the \( P \)-specific quasi-equilibrium, the effort level of each player in \( P \) is the best response to the other players' effort levels — it satisfies either (2) or (3), given the other players' effort levels.

**Lemma 4.** (a) Given \( Y \), if \( X^b(Y; 1) = 0 \), then there is a unique \( P \)-specific quasi-equilibrium at which all the players in \( P \) expend zero effort. (b) If \( X^b(Y; 1) > 0 \) and \( (v_1^A - v_1^B) > (v_2^A - v_2^B) \), then there is a unique \( P \)-specific quasi-equilibrium at which \( x_1^* = X^b(Y; 1) \) and \( x_s^* = 0 \) for \( s = 2, \ldots, k \). (c) If \( X^b(Y; 1) > 0 \) and \( (v_1^A - v_1^B) = (v_h^A - v_h^B) > (v_{h+1}^A - v_{h+1}^B) \) for some \( h \), then there are multiple \( P \)-specific quasi-equilibria at which \( \sum_{i=1}^h x_i^* = X^b(Y; 1) \) and \( x_s^* = 0 \) for \( s = h+1, \ldots, k \), where \( 2 \leq h \leq k \).

The proof of Lemma 4 is provided in Appendix D. Part (a) refers to the case where a given effort level of the players in \( N \) is "significantly" large relative to player 1's valuation spread (or stake), whereas parts (b) and (c) refer to the opposite case. Part (b) says that if just one player in \( P \) has the widest valuation spread (or the highest stake), there is a unique \( P \)-specific quasi-equilibrium at which only that player, player 1, expends positive effort — more specifically, the optimal effort level for himself — and the rest expend zero effort. Part (c) says that if more than one player in \( P \) has the widest valuation spread (or the highest stake), there are multiple \( P \)-specific quasi-equilibria. At these quasi-equilibria, if a player expends positive effort, then he must have the widest positive valuation spread (or the highest stake); however,
there may be players with the widest positive valuation spread (or the highest stake) who expend zero effort.

Lemma 4 can be explained as follows. Because the players with the widest positive valuation spread (or the highest stake) have the greatest gross marginal payoff at any effort level, and because all the players have the same marginal cost, at the effort level optimal for the players with the widest positive valuation spread (or the highest stake), the gross marginal payoff for each of the other players in $\mathbb{P}$ is less than the marginal cost. This implies that, at the effort level optimal for the players with the widest positive valuation spread (or the highest stake), the players expending positive effort except the players with the widest positive valuation spread (or the highest stake) — have an incentive to decrease their effort levels. Another fact is that, at the effort level at which the gross marginal payoff for the players with the widest positive valuation spread (or the highest stake) is not equal to the marginal cost, the players with the widest positive valuation spread (or the highest stake) have an incentive to change their effort levels. All this implies that a quasi-equilibrium does not exist unless positive effort is exerted only by players with the widest positive valuation spread (or the highest stake).

3.2. $\mathbb{N}$-specific quasi-equilibria

Consider the players in $\mathbb{N}$, players $k+1$ through $n$, each of whom has a negative valuation spread. For $j \in \mathbb{N}$, the negative of player $j$’s valuation spread, $(v_j^B - v_j^A)$, is called his stake. Based on Lemma 1, I restrict attention only to the strategy profiles at which $y_i = 0$ for $i \in \mathbb{P}$ and $x_j = 0$ for $j \in \mathbb{N}$. Thus, as in Section 3.1, $X$ is the effort level expended for prize $A$ only by the players in $\mathbb{P}$, and $Y$ is the effort level expended for prize $B$ only by the players in $\mathbb{N}$. For $j \in \mathbb{N}$, let $y_j^B$ represent player $j$’s best response to the other players’ effort levels (including those expended for prize $A$ by the players in $\mathbb{P}$). Then, because $y_j^B$ is player $j$’s effort level expended for prize $B$ which maximizes his expected payoff

$$\pi_j = v_j^B + (v_j^A - v_j^B) p^A(X, Y) - y_j$$
subject to $y_j \geq 0$, it satisfies the following first-order condition:

$$(v_j^A - v_j^B)(\partial p^A / \partial y_j) - 1 = (v_j^A - v_j^B)(\partial p^A / \partial Y) - 1 = 0 \quad \text{for } y_j^A > 0$$

(7)

or

$$(v_j^A - v_j^B)(\partial p^A / \partial y_j) - 1 = (v_j^A - v_j^B)(\partial p^A / \partial Y) - 1 \leq 0 \quad \text{for } y_j^A = 0.$$  

(8)

In the case where $y_j^A > 0$, player $j$'s marginal gross payoff, $(v_j^A - v_j^B)(\partial p^A / \partial y_j)$, is equal to his marginal cost, 1, at the positive effort level, $y_j^A$. In the case where $y_j^A = 0$, his marginal gross payoff does not exceed his marginal cost at the zero effort levels. Because player $j$'s marginal gross payoff, $(v_j^A - v_j^B)(\partial p^A / \partial y_j)$, decreases in his effort level, $y_j$, due to Assumption 2, his payoff function is strictly concave in his effort level. This implies that the second-order condition for maximizing $\pi_j$ is satisfied, and also that $y_j^A$ is unique.

Define an $\mathbb{N}$-specific quasi-equilibrium, denoted by $(y_{k+1}^*, \ldots, y_n^*)$, as an $(n-k)$-tuple vector of effort levels expended for prize $B$, one for each player in $\mathbb{N}$, at which $y_j^*$, for $j \in \mathbb{N}$, is the best response to the other players' effort levels (including those expended for prize $A$ by the players in $\mathbb{P}$). To obtain $\mathbb{N}$-specific quasi-equilibria, I begin by defining the optimal effort level for player $j$ given the effort level $X$ of the players in $\mathbb{P}$, where $j \in \mathbb{N}$.

**Definition 2.** For $j \in \mathbb{N}$, let $Y^b(X; j)$ denote the optimal effort level for player $j$ given the effort level $X$ of the players in $\mathbb{P}$ (or, for short, player $j$'s optimal effort given $X$). I define $Y^b(X; j)$ as the effort level that maximizes

$$v_j^B + (v_j^A - v_j^B) p^A(X, Y) - Y$$

subject to $Y \geq 0$.

(9)

Player $j$'s optimal effort given $X$ can be considered as the best response of the set $\mathbb{N}$ of players as a whole to the effort level $X$ of the players in $\mathbb{P}$ when $v_j^A$ and $v_j^B$ are taken as $\mathbb{N}$'s valuations for prizes $A$ and $B$, respectively. Simply speaking, it represents $\mathbb{N}$'s best response to $X$
which is computed using player $j$'s valuations for the prizes. Accordingly, $Y^b(X; j)$ satisfies the following first-order condition:

$$(v^A_j - v^B_j)(\partial p^A / \partial Y) - 1 = 0 \quad \text{for } Y^b(X; j) > 0 \quad (10)$$

or

$$(v^A_j - v^B_j)(\partial p^A / \partial Y) - 1 \leq 0 \quad \text{for } Y^b(X; j) = 0. \quad (11)$$

Because, under Assumption 2, $(v^A_j - v^B_j)(\partial p^A / \partial Y)$ in (10) and (11) decreases in $Y$, (9) is strictly concave in $Y$. This implies that the second-order condition for maximizing (9) is satisfied, and also that $Y^b(X; j)$ is unique.

Next, in Lemma 5, I compare $Y^b(X; h)$ with $Y^b(X; t)$, for $h, t \in \mathbb{N}$ with $h \neq t$. In other words, I compare $\tilde{\mathbb{N}}$'s best responses to $X$, each computed using a different player's valuations for prizes $A$ and $B$.

**Lemma 5.** Given the effort level $X$, $\tilde{\mathbb{N}}$'s best response computed using a wider negative valuation spread (or a higher stake) is greater than or equal to that computed using a narrower negative valuation spread (or a lower stake): $Y^b(X; s) \geq Y^b(X; s - 1)$ for $s = k + 2, \ldots, n$.

Lemma 5 says that, given the effort level $X$ of the players in $\tilde{\mathbb{P}}$, $Y^b(X; n)$ is the largest, $Y^b(X; k + 1)$ is the smallest, and the rest are in between.

Next, in Lemma 6, I demonstrate that the effort level of the players in $\mathbb{N}$, $Y^* \equiv \sum_{j=k+1}^{n} y_j^*$, at an $\mathbb{N}$-specific quasi-equilibrium given the effort level $X$ must be equal to player $n$'s optimal effort given $X$: $Y^* = Y^b(X; n)$. This result is useful in obtaining $\mathbb{N}$-specific quasi-equilibria given $X$ in Lemma 7.

**Lemma 6.** The effort level $Y^*$ of the players in $\mathbb{N}$ at an $\mathbb{N}$-specific quasi-equilibrium given $X$ is equal to $Y^b(X; n)$. 
Lemma 6 says that, given an effort level of the players in $\mathbb{P}$, the effort level of the players in $\mathbb{N}$ at an $\mathbb{N}$-specific quasi-equilibrium is exactly the same as the optimal effort level for player $n$, the player with the widest negative valuation spread (or the highest stake). Consequently, the effort level of the players in $\mathbb{N}$ at an $\mathbb{N}$-specific quasi-equilibrium given $X$ depends only on the widest of the valuation spreads (or the highest of the stakes) that the players in $\mathbb{N}$ have.

Finally, in Lemma 7, I obtain $\mathbb{N}$-specific quasi-equilibria given $X$. Recall that an $\mathbb{N}$-specific quasi-equilibrium given $X$ is defined as an $(n-k)$-tuple vector, $(y_{k+1}^*, \ldots, y_n^*)$, of effort levels expended for prize $B$, one for each player in $\mathbb{N}$. At the $\mathbb{N}$-specific quasi-equilibrium, the effort level of each player in $\mathbb{N}$ is the best response to the other players' effort levels — it satisfies either (7) or (8), given the other players' effort levels.

**Lemma 7.** (a) Given $X$, if $Y^b(X; n) = 0$, then there is a unique $\mathbb{N}$-specific quasi-equilibrium at which all the players in $\mathbb{N}$ expend zero effort. (b) If $Y^b(X; n) > 0$ and $(v_{n-1}^A - v_{n-1}^B) > (v_n^A - v_n^B)$, then there is a unique $\mathbb{N}$-specific quasi-equilibrium at which $y_s^* = Y^b(X; n)$ and $y_s^* = 0$ for $s = k+1, \ldots, n-1$. (c) If $Y^b(X; n) > 0$ and $(v_{i-1}^A - v_{i-1}^B) > (v_i^A - v_i^B) = (v_n^A - v_n^B)$ for some $t$, then there are multiple $\mathbb{N}$-specific quasi-equilibria at which $\sum_{i=t}^{n} y_i^* = Y^b(X; n)$ and $y_s^* = 0$ for $s = k+1, \ldots, t-1$, where $k+1 \leq t \leq n-1$.

Part (b) says that if just one player in $\mathbb{N}$ has the widest valuation spread (or the highest stake), there is a unique $\mathbb{N}$-specific quasi-equilibrium at which only that player, player $n$, expends positive effort and the rest expend zero effort. Part (c) says that if more than one player in $\mathbb{N}$ has the widest valuation spread (or the highest stake), there are multiple $\mathbb{N}$-specific quasi-equilibria. At these quasi-equilibria, if a player expends positive effort, then he must have the widest negative valuation spread (or the highest stake); however, there may be players with the widest negative valuation spread (or the highest stake) who expend zero effort.

The explanation for Lemma 7 can be made similarly to that for Lemma 4.
3.3. Pure-strategy Nash equilibria of the game

Now, in the Theorem, I obtain the pure-strategy Nash equilibria of the game. At a Nash equilibrium, each player's pair of effort levels, one for prize $A$ and the other for prize $B$, is the best response to the other players' pairs of effort levels. Denote a pure-strategy Nash equilibrium by a $2n$-tuple vector of effort levels, $(x_1^N, y_1^N, \ldots, x_n^N, y_n^N)$. Let $X^N \equiv \sum_{i=1}^{k} x_i^N$ and let $Y^N \equiv \sum_{j=k+1}^{n} y_j^N$.

A pure-strategy Nash equilibrium occurs if and only if $y_i^N = 0$ for $i \in \mathbb{P}$, the set $\mathbb{P}$ is in $\mathbb{P}$-specific quasi-equilibrium, $x_j^N = 0$ for $j \in \mathbb{N}$, and the set $\mathbb{N}$ is in $\mathbb{N}$-specific quasi-equilibrium. Thus, using Lemmas 1, 4, and 7, it is straightforward to obtain the following theorem.

**Theorem.** (a) If $(v_1^A - v_1^B) > (v_2^A - v_2^B)$ and $(v_{n-1}^A - v_{n-1}^B) > (v_n^A - v_n^B)$,
then there is a unique pure-strategy Nash equilibrium at which $(x_1^N, y_1^N) = (X^b(Y^N; 1), 0)$, $(x_n^N, y_n^N) = (0, Y^b(X^N; n))$, and $(x_s^N, y_s^N) = (0, 0)$ for $s = 2, \ldots, n-1$.

(b) If $(v_1^A - v_1^B) > (v_2^A - v_2^B)$ and $(v_{t-1}^A - v_{t-1}^B) > (v_t^A - v_t^B)$ for some $t$,
then there are multiple Nash equilibria at which $(x_1^N, y_1^N) = (X^b(Y^N; 1), 0)$, $\sum_{j=t}^{n} x_j^N = 0$,
$\sum_{j=t}^{n} y_j^N = Y^b(X^N; n)$, and $(x_s^N, y_s^N) = (0, 0)$ for $s = 2, \ldots, t-1$, where $k+1 \leq t \leq n-1$.

(c) If $(v_1^A - v_1^B) = (v_h^A - v_h^B) > (v_{h+1}^A - v_{h+1}^B)$ for some $h$ and $(v_{n-1}^A - v_{n-1}^B) > (v_n^A - v_n^B)$,
then there are multiple Nash equilibria at which $\sum_{i=1}^{h} x_i^N = X^b(Y^N; 1)$, $\sum_{i=1}^{h} y_i^N = 0$, $(x_n^N, y_n^N) = (0, Y^b(X^N; n))$, and $(x_s^N, y_s^N) = (0, 0)$ for $s = h+1, \ldots, n-1$, where $2 \leq h \leq k$.

(d) If $(v_1^A - v_1^B) = (v_h^A - v_h^B) > (v_{h+1}^A - v_{h+1}^B)$ for some $h$ and $(v_{t-1}^A - v_{t-1}^B) > (v_t^A - v_t^B)$
$=(v_n^A - v_n^B)$ for some $t$, then there are multiple Nash equilibria at which $\sum_{i=1}^{h} x_i^N = X^b(Y^N; 1)$,
$\sum_{i=1}^{h} y_i^N = 0$, $\sum_{j=t}^{n} x_j^N = 0$, $\sum_{j=t}^{n} y_j^N = Y^b(X^N; n)$, and $(x_s^N, y_s^N) = (0, 0)$ for $s = h+1, \ldots, t-1$, where $2 \leq h \leq k$ and $k+1 \leq t \leq n-1$. 
Part (a) of the Theorem says that, if there are just one player with the widest positive valuation spread and just one player with the widest negative valuation spread, then there exists a unique Nash equilibrium at which only the two players, player 1 and player $n$, expend positive effort. It says also that, at this equilibrium, player 1's effort level expended for prize $A$, $x_1^N$, is the best response to player $n$'s effort level expended for prize $B$, $y_n^N$, and vice versa. On the other hand, parts (b) through (d) say that, if there is more than one player with the widest positive valuation spread or more than one player with the widest negative valuation spread, then there exist multiple Nash equilibria. In this case, there are equilibria at which some, but not all, players with the widest positive [negative] valuation spread expend zero effort. Note, however, that the equilibrium effort level expended for prize $A$ is always equal to the effort level which is optimal from player 1's perspective, given the equilibrium effort level expended for prize $B$. Note also that the equilibrium effort level expended for prize $B$ is always equal to the effort level which is optimal from player $n$'s perspective, given the equilibrium effort level expended for prize $A$.

Another interesting observation from the Theorem is that the equilibrium effort level expended for prize $A$, that expended for prize $B$, and the equilibrium total effort level are equal to those obtained in the reduced contest in which only player 1 (with the widest positive valuation spread) and player $n$ (with the widest negative valuation spread) compete. This observation suggests that one may obtain those equilibrium effort levels simply by solving the reduced two-player game.

To highlight interesting and important results obtained at the pure-strategy Nash equilibria of the game, I summarize them in the following remark.

**Remark.** (a) Only the players' valuation spreads (or stakes) matter in equilibrium: The players' equilibrium effort levels do not depend separately on their valuations for prize $A$ and their valuations for prize $B$, but solely on their valuation spreads (or stakes). (b) In equilibrium, the players with a nonnegative valuation spread expend zero effort for prize $B$; the players with a
negative valuation spread expend zero effort for prize A. (c) In equilibrium, the player or players with the widest positive valuation spread expend positive effort for prize A, the player or players with the widest negative valuation spread expend positive effort for prize B, and a player whose valuation spread is narrower than somebody else's expends zero effort for both prizes. 

(d) The equilibrium effort level expended for each prize and the equilibrium total effort level are determined only by the widest positive valuation spread and the widest negative valuation spread. (e) The equilibrium expected payoff for an effort-expending player may be less than the equilibrium expected payoff for a free rider.

It follows from part (c) of the Remark that, in equilibrium, there may be three distinct "groups": the players who expend positive effort only for prize A, the players who expend positive effort only for prize B, and the players who expend zero effort for both prizes. However, the groups are only identified, but not formally formed. Clearly, part (c) suggests that the free-rider problem is prevalent and severe. I am tempted to say that "moderate players" free ride on extremists' contributions. Interestingly, part (c) implies that a player with the highest valuation for prize A or prize B may expend zero effort; furthermore, even a player with the highest positive valuation for every prize, if any, does not expend any effort for either prize if his valuation spread, positive or negative, is narrower than somebody else's. On the other hand, part (c) implies that a player with negative valuations for both prizes — in this case, the prizes are public goods for him — may expend positive effort for either prize.

4. Conclusions

I have studied a contest in which n risk-neutral players compete, by expending irreversible effort, over which one of alternative prizes to have awarded to them by the decision maker. The alternative prizes are public goods/bads for the players. The players' valuations for the prizes are publicly known. The players have no budget constraints, and choose their effort levels simultaneously and independently.
I have obtained the pure-strategy Nash equilibria of the game, and established the following important results. First, the players' equilibrium effort levels do not depend separately on their valuations for prize $A$ and their valuations for prize $B$, but only on their valuation spreads between prizes $A$ and $B$. Second, in equilibrium, a player with preference for prize $A$ over prize $B$ never expends positive effort for prize $B$; a player with the opposite preference never expends positive effort for prize $A$. Third, in equilibrium, the free-rider problem is prevalent and severe. Indeed, in equilibrium, only the player or players with the widest positive valuation spread and the player or players with the widest negative valuation spread expend positive effort. Finally, the equilibrium effort level expended for each prize and the equilibrium total effort level are determined only by these two values: the widest positive valuation spread and the widest negative valuation spread.

In this paper, I have assumed that the players have no budget constraints. What happens if the players are assumed to be budget-constrained? One can adapt the analysis and results in Section 4 of Baik (2008) to the modified model with budget-constrained players, and obtain, for example, the following. First, if the total budget of the players in $\mathbb{P}$ does not exceed the optimal effort level for player $k$ given the equilibrium effort level of the players in $\mathbb{N}$, $X^b(Y^N; k)$, then all the players in $\mathbb{P}$ exhaust their budgets. Similarly, if the total budget of the players in $\mathbb{N}$ does not exceed player $(k+1)$'s optimal effort given $X^N$, $Y^b(X^N; k+1)$, then all the players in $\mathbb{N}$ exhaust their budgets. Second, if a player in $\mathbb{P}$ [in $\mathbb{N}$] expends positive effort, then the players in $\mathbb{P}$ [in $\mathbb{N}$] who have higher stakes than the player exhaust their budgets. Third, if a player in $\mathbb{P}$ [in $\mathbb{N}$] expends zero effort, then the players in $\mathbb{P}$ [in $\mathbb{N}$] who have lower stakes than the player expend zero effort.

I have assumed that each player's marginal cost of increasing effort is constant. Is the free-rider problem alleviated if each player's marginal cost is increasing? The answer seems to be yes. Consider, for example, the players in $\mathbb{P}$. Suppose that only player 1 expends a positive effort level, which is equal to player 1's optimal effort given $Y$, $X^b(Y; 1)$. Since the players' marginal costs are increasing, the gross marginal payoffs at the effort level for some players in $\mathbb{P}$
(including player 2) may be greater than their marginal costs at the zero effort levels, and if so, those players have an incentive to increase their effort levels. This implies that, if each player's marginal cost is increasing, then there may be more players expending positive effort in equilibrium, as compared with the current model in which each player's marginal cost is constant. However, it appears that the modified game with increasing marginal costs is not tractable for characterizing its Nash equilibria.

I have assumed that there are two public-good/bad prizes. It would be interesting to study an extended model in which there are more than two public-good/bad prizes. In fact, I have been working on a model in which there are three public-good/bad prizes. A preliminary analysis shows that, given three prizes, $A$, $B$, and $C$, three sets of players may be endogenously identified in the society: the set $Z_A$ of players who least prefer prize $A$, the set $Z_B$ of players who least prefer prize $B$, and the set $Z_C$ of players who least prefer prize $C$. It shows also that, in equilibrium, player $i$, for $i \in Z_A$, expends zero effort for prize $A$; player $j$, for $j \in Z_B$, expends zero effort for prize $B$; and player $k$, for $k \in Z_C$, expends zero effort for prize $C$. However, because of analytical complexity involved, I may not complete the analysis.

In this paper, I have assumed that the players' valuations for the prizes are publicly known. It would be interesting to study an extended model in which the players have incomplete information about their valuations for the prizes. Another possible extension of this paper would be to consider a model in which alternative prizes or policies are endogenized (see Epstein and Nitzan 2004). Finally, it would be interesting to study modified models in which the selection probability functions for the prizes are different from those in the current model. I leave these extensions and modifications for future research.
Footnotes


2. It is easy to see that a player with zero valuation spreads expends zero effort for both prizes in equilibrium.

3. Since a player with zero valuation spreads expends zero effort for both prizes in equilibrium, I may ignore his presence for concise exposition.

4. It would be natural to define a $\mathbb{P}$-specific quasi-equilibrium as a $2k$-tuple vector of effort levels because each player chooses two effort levels, one for prize $A$ and the other for prize $B$. However, since all the $k$ players in $\mathbb{P}$ expend zero effort for prize $B$, I define it, for simplicity, only with the players’ effort levels expended for prize $A$.

5. Note that if $(v_2^A - v_2^B) < 0$, then only player 1 belongs to $\mathbb{P}$ and thus $k = 1$.

6. The proofs and explanations of lemmas in Section 3.2 are similar to those in Section 3.1, and therefore partly omitted.

7. Note that if $(v_{n-1}^A - v_{n-1}^B) \geq 0$, then only player $n$ belongs to $\mathbb{N}$.

8. I assume here that $X^b(Y^N; 1)$ and $Y^b(X^N; n)$ are both positive.

9. In the case where $X^b(Y^N; 1) > 0$, at least one of the players with the widest positive valuation spread expends positive effort for prize $A$. In the case where $Y^b(X^N; n) > 0$, at least one of the players with the widest negative valuation spread expends positive effort for prize $B$. 
Appendix A: Proof of Lemma 1

(a) Consider a strategy profile, \((x_1, y_1, \ldots, x_n, y_n)\). From (1), the expected payoff for player \(i\) is
\[ \pi_i = y_i^B + (y_i^A - y_i^B) p^A(X, Y) - x_i - y_i. \]
Suppose that player \(i\), for \(i \in I\), expends positive effort of \(y_i\) for prize \(B\). Then, since \((y_i^A - y_i^B) \geq 0\) holds, it follows from Assumption 2 that, ceteris paribus, the expected payoff for player \(i\) increases when \(y_i\) decreases. This means that player \(i\), for \(i \in I\), has an incentive to decrease his effort level for prize \(B\) all the way down to zero.

(b) The proof of part (b) is similar to that of part (a), and therefore omitted.

Appendix B: Proof of Lemma 2

It is trivial to see that for any \(h, t \in I\), \(X^b(Y; h) = X^b(Y; t)\) holds if \((y_h^A - y_h^B) = (y_t^A - y_t^B)\). Next, consider the case where \((y_h^A - y_h^B) > (y_t^A - y_t^B)\) holds for \(h, t \in I\) with \(h \neq t\). Since the term, \((y_h^A - y_h^B) (\partial p^A / \partial X)\), in (5) and (6) decreases in \(X\), it is straightforward to obtain that for any \(h\) and \(t\), \(X^b(Y; h) = X^b(Y; t) = 0\) holds if \(X^b(Y; h) = 0\), and \(X^b(Y; h) > X^b(Y; t)\) holds if \(X^b(Y; h) > 0\). The above together with Assumption 1 yields Lemma 2.
Appendix C: Proof of Lemma 3

Suppose that the effort level $X$ of the players in $\mathbb{P}$ is less than $X^b(Y; 1)$. Then, since $X$ is less than the optimal effort level for player 1, player 1 has an incentive to increase his effort level.

Next, suppose that the effort level $X$ of the players in $\mathbb{P}$ is greater than $X^b(Y; 1)$. Then, due to Lemma 2, $X > X^b(Y; 1) \geq X^b(Y; h)$ holds for any $h \in \mathbb{P}$. It follows from this and Assumption 2 that $(v^A_i - v^B_i)(\partial p^A_i / \partial X) - 1 < 0$ holds at $X$ for any $h \in \mathbb{P}$ (see (5) and (6)). In this case, any player expending a positive effort level has an incentive to decrease his effort level since his marginal gross payoff is less than the constant marginal cost of 1 (see (2) and (3)).

Appendix D: Proof of Lemma 4

(a) If $X^b(Y; 1) = 0$, then by Lemma 2, $X^b(Y; i) = 0$ holds for any $i \in \mathbb{P}$. Then it follows from (6) that if all the players in $\mathbb{P}$ expend zero effort, then $(v^A_i - v^B_i)(\partial p^A_i / \partial X) - 1 \leq 0$ holds for any $i \in \mathbb{P}$. This implies that if all the players in $\mathbb{P}$ expend zero effort, then (3) is satisfied for each player in $\mathbb{P}$. Therefore, given $Y$ and zero effort levels of the other players in $\mathbb{P}$, zero effort is the best response of each player in $\mathbb{P}$. Next, the uniqueness of the $\mathbb{P}$-specific quasi-equilibrium follows immediately from Lemma 3 and the fact that effort levels are nonnegative.

(b) Using (6) and Assumption 1, I obtain: If the effort level of the players in $\mathbb{P}$ is equal to $X^b(Y; 1)$, then $(v^A_i - v^B_i)(\partial p^A_i / \partial X) - 1 = 0$ and $(v^A_s - v^B_s)(\partial p^A_s / \partial X) - 1 < 0$ for $s = 2, \ldots, k$ hold. Hence, if player 1 expends $X^b(Y; 1)$ and the rest of the players in $\mathbb{P}$ choose zero effort levels, then (2) is satisfied for player 1, and (3) is satisfied for each of the rest. Therefore, the proposed effort level of each player in $\mathbb{P}$ is his best response to $Y$ and the proposed effort levels of the other players in $\mathbb{P}$.
Next, I show that no other $k$-tuple vector of effort levels (of the players in $\mathbb{P}$) constitutes a $\mathbb{P}$-specific quasi-equilibrium. First, it follows from Lemma 3 that a vector of effort levels, $(x_1, \ldots, x_k)$, does not constitute a $\mathbb{P}$-specific quasi-equilibrium if $\sum_{i=1}^{k} x_i \neq X^b(Y; 1)$. Second, suppose on the contrary that a vector of effort levels, $(x_1, \ldots, x_k)$, constitutes a $\mathbb{P}$-specific quasi-equilibrium, where $\sum_{i=1}^{k} x_i \equiv X = X^b(Y; 1)$ and $x_i > 0$ for some $t$, where $2 \leq t \leq k$. Then because (2) must be satisfied for player $t$, at the vector of effort levels $(v_t^A - v_t^B)(\partial p^A / \partial X) - 1 = 0$ must hold. On the other hand, it follows from (5) that $(v_1^A - v_1^B)(\partial p^A / \partial X) - 1 = 0$ must hold at the vector of effort levels because $X = X^b(Y; 1) > 0$. This implies that $(v_t^A - v_t^B)(\partial p^A / \partial X) - 1 < 0$ must hold because $(v_t^A - v_t^B) > (v_2^A - v_2^B) \geq (v_1^A - v_1^B)$ due to the conditional statement of part (b) and Assumption 1. This leads to a contradiction.

(c) The proof of part (c) is similar to that of part (b), and therefore omitted.
References


