Semigroups generated by positive semidefinite matrices

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Theorem [Ballantine,1970]

Every $A \in \mathcal{M}_n$ with $\det(A) > 0$ is a product of k matrices in $\mathcal{P}_n$ with $k \leq 5$.

Furthermore, he characterized matrices $A \in \mathcal{M}_n$ with $\det(A) \geq 0$ that can be written as the product of k (invertible) matrices in $\mathcal{P}_n$ for $k = 1, 2, 3, 4, 5$.

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Remark It is interesting that techniques in canonical forms, numerical ranges, integral equations in operators, etc. were used in the study.

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- Furthermore, he conjectured that for $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \mathcal{M}_n$ such that A_1 is invertible and A_3 is a nilpotent, the matrix A is the product of 3 matrices in $\mathcal{P}$ if and only if A_1 is the product of 3 positive definite matrices.

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- For example, a matrix $A \in \mathcal{M}_n$ satisfies $A = PQ$ with $P, Q \in \mathcal{P}_n$ if and only if it is similar to a nonnegative diagonal matrix.
- Equivalently, the set

$$\{X \in \mathcal{P}_n : \det(X) > 0, AX = XA^*\}$$

is congruent to a geodesic submanifold of the set of positive definite matrices, as shown in a paper of Lim in 2011.

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- The inclusion is proper.
- Example: $A = -I \in \mathcal{S}_1 \cap \mathcal{S}_2$, but $A \notin \mathcal{S}_3$.

- It is known that if $A \in \mathcal{M}_n$ is the product of two contractions in $\mathcal{P}_n$, then

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Then A is the product of two positive semidefinite contractions if and only if $a, b \in [0, 1]$ and

$$\|P\| \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}.$$

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- That is, if A in the closure of the set of finite product of positive semidefinite contractions, is it a product of a finite product of positive semidefinite contractions.

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- What if A_0 is invertible? Say, $A_0 = \mathrm{diag}\,(-0.9i, 0.8i)$.

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- Every real matrix $A \in \mathcal{M}_n$ with $\det(A) \geq 0$ is the product of k real matrices in $\mathcal{P}_n$ with $k \leq 5$.
- Let $A \in \mathcal{M}_n(\mathbb{R})$. Then $A = PQ$ for two real matrices P and Q in $\mathcal{P}_n$ if and only if A is similar to a nonnegative diagonal matrix.
- For example, let $A = -I_2$. Then

$$\begin{pmatrix} 2 & \\ & 4 \end{pmatrix} A = \begin{pmatrix} -2 & \\ & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} -4 & 4 \\ -3 & 2 \end{pmatrix} = P_1 P_2 P_3 P_4.$$

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Extensions to infinite dimensional operators

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Thank you.

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