Product of two positive contractions

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$A, B \in M_n = \{$ all $n \times n$ complex matrices$\}$

Definition

(1) A is positive if $\langle Ax, x\rangle > 0$ for all $x \neq 0$.

$$x = [x_1 \dots x_n]^T,\ y = [y_1 \dots y_n]^T$$

$$\Rightarrow \langle x, y\rangle = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n,\ \|x\| = \sqrt{\langle x, x\rangle}$$

(2) A is positive semidefinite (PSD) if $\langle Ax, x\rangle \geq 0$ for all x.

(3) $A \geq B$ if $A - B$ is PSD

(4) $\sigma(A) = \{\lambda : Ax = \lambda x, x \neq 0\}$: spectrum of A.

(5) A is contraction if $\sup_{\|x\|=1} \|Ax\| \leq 1$.

Proposition

(1) $A \in M_n$ is PSD (resp. positive) $\Leftrightarrow$ $A \cong \text{diag}(a_1, \ldots, a_n)$ $\forall\, a_i \geq 0$(resp. > 0).

(2) A, B are PSD $\Rightarrow \sigma(AB) \subseteq [0, \infty)$

(3) A, B are PSD contractions $\Rightarrow \sigma(AB) \subseteq [0, 1]$

pf.

(2)

$$\sigma(AB) = \sigma(A^{1/2}A^{1/2}B) = \sigma(A^{1/2}BA^{1/2}) \subseteq [0, \infty)$$

Theorem(1988,Wu)

$T = AB$, where A, B are PSD $\Leftrightarrow$ T is similar to a nonnegative diagonal matrix.

Example Let $A = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$, $a > b > 0, c > 0$

We have

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \begin{pmatrix} \frac{a-b}{c} & 1 \\ 1 & d \end{pmatrix} = \begin{pmatrix} a\frac{a-b}{c} & a \\ a & c+bd \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} a\frac{a-b}{c} & a \\ a & c+bd \end{pmatrix} \begin{pmatrix} \frac{a-b}{c} & 1 \\ 1 & d \end{pmatrix}^{-1}$$

Note:

A is the product of two PSD contractions $\Rightarrow$ A is contraction similar to a nonnegative diagonal matrix.

The converse is not true.

Counterexample(2014, Li and Tsai)

Let $A = \frac{1}{25}\begin{pmatrix} 9 & 3 \\ 0 & 16 \end{pmatrix}$. A is contraction similar to $\operatorname{diag}(9/25, 16/25)$ but cannot be written as a product of two PSD contractions.

Theorem(2014, Li and Tsai)

$A = \begin{pmatrix} a & p \\ 0 & b \end{pmatrix}$ is the product of two PSD contractions if and only if $a, b \in [0, 1]$ and $|p| \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}$.

Note:

The result can be extend to a quadratic operator on a complex Hilbert space.

Question.

How to characterize a square matrix that can be written as the product of two PSD contractions?

Recall:

A is orthogonal projection if A is PSD with $\sigma(A) \subseteq \{0, 1\}$.

Proposition

A is the product of two orthogonal projections if and only if

$A \cong I_p \oplus 0_q \oplus A_1 \oplus \cdots \oplus A_m$, where

$$A_i = \begin{pmatrix} \lambda_i & \sqrt{\lambda_i - \lambda_i^2} \\ 0 & 0 \end{pmatrix} \quad 0 < \lambda_i < 1 \text{ for all } i = 1, \ldots, m$$

Proposition

Suppose that $A \in M_n$ is the product of two PSD contractions. Then

$$A \cong I_p \oplus \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0_{n-p-m} \end{pmatrix},$$

where $A_{11} \in M_m$ is similar to $\operatorname{diag}(a_1, \ldots, a_m)$ with $a_i \in (0,1)\ \forall i$.

pf.

A is the product of two PSD contractions

$\Rightarrow$ (1) A is contraction, (2) $\sigma(A) \subseteq [0,1]$, (3) A is similar to a diagonal matrix

(2) $\Rightarrow A \cong \begin{pmatrix} I_p & B_1 & B_2 \\ 0 & A_{11} & A_{12} \\ 0 & 0 & A_{22} \end{pmatrix}$, where $\sigma(A_{11}) \subseteq (0,1)$ and $\sigma(A_{22}) = \{0\}$.

(1) $\Rightarrow B_1 = 0, B_2 = 0$

(3) $\Rightarrow A_{11}, A_{22}$ are similar to diagonal matrices and so $A_{22} = 0$.

Theorem

Suppose that

$$A = I_p \oplus \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0_{n-p-m} \end{pmatrix},$$

where $A_{11} \in M_m$ is similar to $D \equiv \mathrm{diag}(a_1, \ldots, a_m)$ with $a_i \in (0,1)\ \forall i$. The following conditions are equivalent.

(1) A is the product of two PSD contractions.

(2) There are matrices $R, C \in M_m$ such that

$$I_p \oplus \begin{pmatrix} A_{11} & A_{12} & 0 & A_{11}C \\ 0 & 0_{n-p-m} & 0 & 0 \\ RA_{11} & RA_{12} & 0_m & RA_{11}C \\ 0 & 0 & 0 & 0_m \end{pmatrix} \in M_{n+2m}$$

is the product of two orthogonal projection.

(3) There is an invertible contraction $U \in M_m$ satisfying

$$A_{11}U = UD \text{ and } UDU^* \geq A_{11}A_{11}^* + A_{12}A_{12}^*$$

Note:

(a) If the condition (3) holds, we have $A = (I_p \oplus P)(I_p \oplus Q)$ for PSD contractions

$$P = \begin{pmatrix} UU^* & 0 \\ 0 & 0_{n-p-m} \end{pmatrix} \text{ and } Q = \begin{pmatrix} (U^*)^{-1}DU^{-1} & (UU^*)^{-1}A_{12} \\ A_{12}^*(UU^*)^{-1} & A_{12}^*(UDU^*)^{-1}A_{12} \end{pmatrix}$$

(b) If $D = \alpha_1 I_{m_1} \oplus \cdots \oplus \alpha_k I_{m_k}$, $1 > \alpha_1 > \cdots > \alpha_k > 0$, then $\exists$ invertible $V = [V_1 \ldots V_k]$ with $V_j^* V_j = I_{m_j}$ such that $V^{-1}A_{11}V = D$.

Hence, $A_{11}U = UD$ if and only if $U = VL$ for some $L = L_1 \oplus \cdots \oplus L_k \in M_{m_1} \oplus \cdots \oplus M_{m_k}$

$$U^*U \leq I_m \Leftrightarrow V^*V \leq (LL^*)^{-1}$$

$$UDU^* \geq A_{11}A_{11}^* + A_{12}A_{12}^* \Leftrightarrow LL^* \leq D^{-1/2}V^{-1}(A_{11}A_{11}^* + A_{12}A_{12}^*)(V^*)^{-1}D^{-1/2}$$

The condition (3) $\Leftrightarrow D^{1/2}V^*(A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}VD^{1/2} \geq (LL^*)^{-1} \geq V^*V$

Theorem

Suppose that

$$A \cong I_p \oplus \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0_{n-p-m} \end{pmatrix},$$

where $A_{11} \in M_m$ is similar to $D \equiv \alpha_1 I_{m_1} \oplus \cdots \oplus \alpha_k I_{m_k}$, $1 > \alpha_1 > \cdots > \alpha_k > 0$. Suppose that $V = [V_1 \ldots V_k]$ with $V_j^* V_j = I_{m_j}$ such that $V^{-1} A_{11} V = D$. The following conditions are equivalent.

(1) A is the product of two PSD contractions.

(2) $\exists\ \Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k \in M_{m_1} \oplus \cdots \oplus M_{m_k}$ satisfying

$$D^{1/2} V^* (A_{11} A_{11}^* + A_{12} A_{12}^*)^{-1} V D^{1/2} \geq \Gamma \geq V^* V$$

Corollary

If A_{11} has distinct eigenvalues, then one only needs to search for a diagonal matrix Γ satisfying the condition (2).

Corollary

For $a, b \in [0,1]$, $A = \begin{pmatrix} a & p \\ 0 & b \end{pmatrix}$ is the product of two PSD contractions if and only if

$$(*) \qquad |p| \leq |\sqrt{a} - \sqrt{b}|\sqrt{(1-a)(1-b)}.$$

pf.

Case 1. a=b: A is the product of two *PSD* contractions $\Rightarrow$ A is similar to a diagonal $\Rightarrow p = 0 \Rightarrow (*)$ holds.

$(*)$ holds. $\Rightarrow p = 0 \Rightarrow A = aI_2$

Case 2. $a \neq b = 1$: $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$.

Case 3. $a \neq b = 0$, $a \neq 1$: $A = \begin{pmatrix} a & p \\ 0 & 0 \end{pmatrix}$ is the product of two *PSD* contractions $\Leftrightarrow |p| \leq \sqrt{a}\sqrt{1-a}$

Case 4. $a \neq b$, $a, b \in (0,1)$: Choose $V = \begin{pmatrix} 1 & p/\mu \\ 0 & (b-a)/\mu \end{pmatrix}$ with $\mu = \sqrt{(a-b)^2 + |p|^2}$. We have

$$AV = V \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \equiv VD \text{ and } V^*V = \begin{pmatrix} 1 & p/\mu \\ p/\mu & 1 \end{pmatrix}.$$

We need to find $\Gamma = \text{diag}(d_1, d_2)$ with $d_1, d_2 \geq 0$ such that

$$D^{1/2}V^*(AA^*)^{-1}VD^{1/2} \geq \Gamma \geq V^*V$$

$$\Leftrightarrow D^{1/2}D^{-1}V^*A^*(A^*)^{-1}A^{-1}AVD^{-1}D^{1/2} \geq \Gamma \geq V^*V$$

$$\Leftrightarrow V^*V - \text{diag}(ad_1, bd_2) \geq 0 \text{ and } \text{diag}(d_1, d_2) - V^*V \geq 0$$

$$\Leftrightarrow d_1 \geq 1 \geq d_1 a,\ (1-d_1 a)(1-d_2 b) \geq |p|^2/\mu^2, \text{ and } (d_1-1)(d_2-1) \geq |p|^2/\mu^2$$

$\Leftrightarrow \exists d_1', d_2' \geq 0$ s.t $(d_1'-1)(d_2'-1) \geq |p|^2/\mu^2$ with $(d_1'-1)(d_2'-1) = (1-d_1'a)(1-d_2'b)$

$\Leftrightarrow \max\limits_{d_1', d_2' \geq 0} (d_1'-1)(d_2'-1) \geq |p|^2/\mu^2$ with $(d_1'-1)(d_2'-1) = (1-d_1'a)(1-d_2'b)$

$\Leftrightarrow (1-a)(1-b)/(1+\sqrt{ab})^2 \geq |p|^2/\mu^2$(by the Langrange multiplier)

$\Leftrightarrow |p|^2 \leq (\sqrt{a}-\sqrt{b})^2(1-a)(1-b)$

Alternating projections

Using alternating projection method to check whether there exists Γ satisfying

$$D^{1/2}V^*(A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}VD^{1/2} \geq \Gamma \geq V^*V$$

Let

$$\Omega_0 = \{\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k \in M_{m_1} \oplus \cdots \oplus M_{m_k} : \Gamma \text{ is PSD}\}$$

$$\Omega_1 = \{\Gamma \in M_m : D^{1/2}V^*(A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}VD^{1/2} \geq \Gamma \geq 0\}$$

and

$$\Omega_2 = \{\Gamma \in M_m : \Gamma \geq V^*V\}$$

For a Hermitian matrix G, $G^+ \equiv (G + \sqrt{G^2})/2$

Proposition

Let $G = [G_{ij}]$ be a Hermitian matrix, where $G_{ii} \in M_{m_i}$.

(1) The projection of G onto Ω_0 is $G_{11}^+ \oplus \cdots \oplus G_{kk}^+$.

(2) The projection of G onto Ω_1 is $X - (X - G)^+$, where
$X = D^{1/2} V^* (A_{11} A_{11}^* + A_{12} A_{12}^*)^{-1} V D^{1/2}$

(3) The projection of G onto Ω_2 is $(G - Y)^+ + Y$, where $Y = V^* V$

We create a sequence $\Gamma_0 \to \hat{\Gamma}_1 \to \Gamma_1 \to \hat{\Gamma}_2 \to \Gamma_2 \to \cdots$. where $\Gamma_k \in \Omega_0, \hat{\Gamma}_{2k-1} \in \Omega_1, \hat{\Gamma}_{2k} \in \Omega_2 \ \forall k \geq 1$. This sequence converges to a solution $\Gamma \in \Omega_0 \cap \Omega_1 \cap \Omega_2$, provided $\Omega_0 \cap \Omega_1 \cap \Omega_2 \neq \emptyset$, see [1].

[1] J.P. Boyle and R.L. Dykstra, Lecture notes in Statistics, Springer, New York (1985), pp. 28–47.

Algorithm

For checking the existence of $\Gamma \in \Omega_0 \cap \Omega_1 \cap \Omega_2$.

Set $k = 0$. Partition X into $[X_{ij}]$ and Y into $[Y_{ij}]$ both conformed to D.

Set $\Gamma_0 = \frac{1}{2}((X_{11} + Y_{11}) \oplus \cdots \oplus (X_{kk} + Y_{kk}))$. Go to Step 1.

Step 1. Change k to $k+1$, and set

$$\hat{\Gamma}_k = \begin{cases} X - (X - \Gamma_{k-1})^+ & \text{if } \ k \text{ is odd,} \\ (\Gamma_{k-1} - Y)^+ + Y & \text{if } \ k \text{ is even.} \end{cases}$$

Partition $\hat{\Gamma}_k$ into $[G_{ij}]$ conformed to D and let $\Gamma_k = G_{11}^+ \oplus \cdots \oplus G_{kk}^+$.

If error$= \max(0, -\lambda_{min}(\Gamma_k - Y)) + \max(0, -\lambda_{min}(X - \Gamma_k)) \approx 0$, stop.

Otherwise, go to step 1.

Note:

Once we have Γ, set $U = V\Gamma^{-1/2}$, we can set $A = (I_p \oplus P)(I_p \oplus Q)$ with

$$P = \begin{pmatrix} UU^* & 0 \\ 0 & 0_{n-p-m} \end{pmatrix} \ \text{and} \ \ Q = \begin{pmatrix} (U^*)^{-1}DU^{-1} & (UU^*)^{-1}A_{12} \\ A_{12}^*(UU^*)^{-1} & A_{12}^*(UDU^*)^{-1}A_{12} \end{pmatrix}$$

Example Let $A = \begin{pmatrix} A_{11} & A_{12} \\ 0_5 & 0_5 \end{pmatrix}$, where

$$A_{11} = \begin{pmatrix} 0.125 & 0.0126 & 0.0033 & 0.024 & -0.0006 \\ 0 & 0.0625 & 0 & 0.012 & 0.0152 \\ 0 & 0 & 0.0625 & 0.0025 & 0.0453 \\ 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.2 \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} 0.0658 & 0.0218 & 0.0031 & 0.05 & -0.0033 \\ 0.0218 & 0.113 & -0.0107 & -0.0120 & 0.0098 \\ 0.0031 & -0.0107 & 0.0418 & 0.0048 & -0.0409 \\ 0.0500 & -0.012 & 0.0048 & 0.1103 & 0.0037 \\ -0.0033 & 0.0098 & -0.0409 & 0.0037 & 0.128 \end{pmatrix}$$

We set

$$V \approx \begin{pmatrix} 1 & -0.1976 & -0.0507 & -0.3169 & -0.0169 \\ 0 & 0.9803 & -0.0102 & -0.0824 & -0.1026 \\ 0 & 0 & 0.9987 & -0.0172 & -0.3108 \\ 0 & 0 & 0 & -0.9447 & 0.0203 \\ 0 & 0 & 0 & 0 & -0.9445 \end{pmatrix}.$$

Using our Matlab program, we obtain

$$\Gamma = \begin{pmatrix} 3.4737 & 0 & 0 & 0 & 0 \\ 0 & 2.3344 & 0.0216 & 0 & 0 \\ 0 & 0.0216 & 2.9472 & 0 & 0 \\ 0 & 0 & 0 & 2.1257 & -0.2132 \\ 0 & 0 & 0 & -0.2132 & 1.6425 \end{pmatrix}.$$

The PSD P and Q defined as the above equation will have error$= \|PQ - A\| = 4.3774x10^{-14}$.

Thanks for your attention!

Example

Let $D = \text{diag}(0.15, 0.15, 0.2)$, $A = \begin{pmatrix} A_{11} & A_{12} \\ 0_3 & 0_3 \end{pmatrix}$ with

$$A_{11} = \begin{pmatrix} 0.15 & 0 & 0 \\ 0 & 0.15 & 0.0375 \\ 0 & 0 & 0.2 \end{pmatrix}$$

and

$$A_{12} = (UDU^* - A_{11}A_{11}^*)^{1/2} = \begin{pmatrix} 0.3571 & 0 & 0 \\ 0 & 0.3215 & 0.1070 \\ 0 & 0.1070 & 0.1689 \end{pmatrix},$$

where

$$U = VR = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5/\sqrt{40} & 3/\sqrt{40} \\ 0 & 0 & 4/\sqrt{40} \end{pmatrix} \text{ with } V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 3/5 \\ 0 & 0 & 4/5 \end{pmatrix}$$

$$\text{and } R = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5/\sqrt{40} & 0 \\ 0 & 0 & 5/\sqrt{40} \end{pmatrix}$$

Then $A_{11}V = VD, A_{11}U = UD$, and U is a contraction such that $UDU^* = A_{11}A_{11}^* + A_{12}A_{12}^*$. Hence, A is the product of two PSD contractions. There is no $\Gamma = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$ such that

$$D^{1/2}V^*(A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}VD^{1/2} = \begin{pmatrix} 1.3 & -0.3 & 0 \\ -0.3 & 1.3 & 0 \\ 0 & 0 & 1.6 \end{pmatrix} \geq \Gamma$$

and

$$\Gamma \geq V^*V = \begin{pmatrix} 1 & 0 & 0.4243 \\ 0 & 1 & -0.4243 \\ 0.4243 & -0.4243 & 1 \end{pmatrix}$$

because $\mu_1, \mu_2 \in (1, 1.3)$ so that $\begin{pmatrix} 1.3-\mu_1 & -0.3 \\ -0.3 & 1.3-\mu_2 \end{pmatrix}$ cannot be PSD.