How to Classify Matrices

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2016 MAO

How to Classify Matrices
by starred equivalence

- ➢ Unitary equivalence
- ➢ *-algebraic isomorphism
- ➢ Complete order isomorphism

in contrast with non-starred cases of

- ❖ Similarity
- ❖ Algebraic isomorphism
- ❖ Linear isometry (To be omitted)
- ❖ Complete isometry (To be omitted)

The equivalent class of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- (1) By similarity

All 2x 2 matrices T of the expression

$$\begin{bmatrix} b & ab \\ -b/a & -b \end{bmatrix} \text{ Or } \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \text{ Or } \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$$

with non-zero a and b.

The equivalent class of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- (2) By Unitary Equivalence

All 2x 2 matrices T of the expression

$$\begin{bmatrix} b & ab \\ -b/a & -b \end{bmatrix} \quad \text{Or} \quad \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} \quad \text{Or} \quad \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$$

with $|b| = |a|/(1+|a|^2)$ and $|c| = 1$.

The equivalent class of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- (3) By Algebraic Equivalence

All n x n matrices T similar to direct sum of k copies of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and p copies of zero.
(Here $k > 0$ and $p \geq 0$ and $n = 2k + p$.)

The equivalent class of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- (4) By *-Algebraic Equivalence

All 2n x 2n matrices T unitarily equivalent to n copies of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

The equivalent class of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- (5) By completely order isomorphism

All n x n matrices T unitarily equivalent to the direct sum of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

and an (n-2) x (n -2) matrix C where the numerical range of C is a subset of the closed disc centred at the origin with radius = ½.

Algebraic Isomorphism

- For each n x n matrix A, there exists a nontrivial polynomial f of lowest degree such that f(A) = 0.

- To classify all n x n matrices by algebraic isomorphism is the same as to write out

  {all complex coefficient polynomials

  degree $\leq$ n with leading coefficient 1}

Jordan Canonical forms

- To classify n x n matrices by similarity is the same as to write out direct sums of Jordan blocks.

- Each building block is a Jordan matrix

$$\begin{bmatrix} \lambda & 1 & 0 & & 0 \\ 0 & \lambda & 1 & & \\ & \ddots & \ddots & \ddots & 0 \\ & & 0 & \lambda & 1 \\ 0 & & & 0 & \lambda \end{bmatrix}$$

depending on the eigenvalue and the size of matrix .

Irreducible matrices

Def. A matrix T is *irreducible* when PT ≠ TP for all orthogonal projections P other than O and I.

- Each matrix is the direct sum of irreducible matrices; where direct summands are unique up to permutation.

- Two irreducible matrices are unitarily eqivalent

iff *-algebraically isomorphic

iff completely order isomorphic.

- A *-homomorphism between two irreducible matrices is always a *-isomorphism.

- Each matrix has unique expression as direct sum of $A_j \otimes I_{mj}$ where I_{mj} is the identity matrix showing the multiplicity m_j; and A_j's are pairwise non-unitarily equivalent irreducible matrices.

- The concern about **multiplicities** shows the precise difference between unitary equivalence and *-algebraic isomorphism.
----the full structure theorems of matrices for unitary equivalence and *-algebraic equivalence

Exercises

(1) Classify all 2 x 2 matrices, up to unitary equivalence.

(2) Classify all 3 x 3 nilpotent irreducible matrices, up to unitary equivalence.

- Classify the complex triples (α, β, γ), as entries in 3 x 3 irreducible matrices of the form

$$\begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix},$$

subjet to unitary equivalence.

Operator Systems

Def. An *operator System* is a self-adjoint linear

subspace of M_n (or B(H)).

- The completely order-isomorphic classification of each matrix T is concerned with the full structure of the operator system in the form of the linear span of {I, T, T*} by means of unital completely positive linear maps.
- Same as to work on the span {I, A, B} where A and B are two hermitian matrices}
- For the non-starred classification, consider the linear span of {I, T} only.

Basic Results about Operator Systems

- **Theorem** (Choi-Effros, 1977) Each injective operator system is completely order isomorphic with a C*-algebra.
- **Corollary** Span {I, T, T*} is injective **iff** the numerical range of T is a 1-point set or a line segment or a triangle.
- Equivalently, **iff** the matrix T is a scalar matrix,
  or a normal matrix with all eigenvalues on a a line,
  or $T = T_1 \oplus T_2$ where T_1 is the diagonal matrix with 3 vertices as diagonal entries and the numerical range of T_2 is subset of the triangle.
- This means that all other matrices are tricky.
- Thus all OTHER 3-dimensionl operator Systems are non-injective. That means that there is no completely positive linear expectation from the full matrix algebra onto the (non-injective) operator system.

Intriguing **Operator Systems**

- **Def**: An Operator System of dimension $\leq$ 3 must be of the form as Span { I, T, T*} with an operator T.

- **Example**: $\left\{\begin{bmatrix} \alpha & \beta \\ \gamma & \alpha \end{bmatrix}\right\}$ is the linear span of

  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ in M_2

- **Example**: {(α, β, γ, δ) with α + γ = β +δ } is a 3-dimensional operator system in the 4-dimensional commutative C*-algebra $\mathbf{C}^4$.

➢ The full structure of unital positive linear maps between these 2 operator systems is beyond imagination.

Intriguing Operator Systems

$$\mathfrak{I} = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \alpha \end{bmatrix} \right\}$$

- In particular, all-positive linear maps from $\mathfrak{I}$ to B(H) are completely positive.

- But, there exist completely positive linear trace-preserving maps from $\mathfrak{I}$ to B(H) which cannot be extendible to trace-preserving completely positive linear maps from M_2 to B(H).

Appendix 1: **Numerical Ranges**

- For an n x n matrix T, the *numerical range*

  $W(T) = \{x^*Tx: x \varepsilon \mathbf{C}^n , \|x\| = 1\}$

  is a convex compact subset of the complex plane.
- **Def**: The *numerical radius*

  $$w(T) = \sup \{|z|: z \text{ in } W(T)\}.$$

- **Problem**: How is $\|T\|$ related to $w(T)$?
- **Standard result**: $\|T\|/2 \leq w(T) \leq \|T\|$.
- **Key Fact**: $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has the SMALLEST

  $w(T) = \|T\| / 2$ BUT the FATTEST $W(T)$

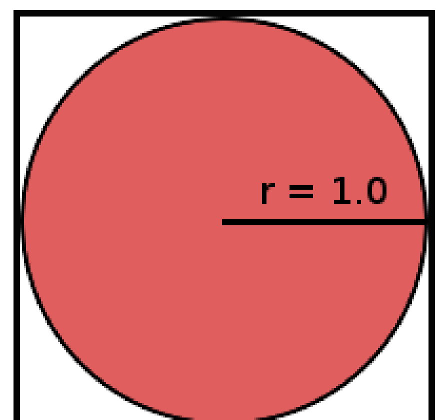
  as the disc centred at 0 .

Unital Positive Linear Maps vs Numerical Ranges

Usually, no way to describe of positive linear maps between 3 x 3 matrices, but there is a graphical way: .

Key Fact: If W(T) is a subset of W(S), then the unital linear map sending I to I and S to T is a positive linear map from the span of {I, S, S*} onto the span of {I,T, T*}.

Example: T= $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and
S = diag { $\sqrt{2}$, i$\sqrt{2}$, - $\sqrt{2}$, -i$\sqrt{2}$} will provide the most intriguing example.



Conclusion:

Classifications of matrices by complete order isomorphisms

- Certainly, we understand all n x n diagonal matrices with $n < 4$.
- We have FULL knowledge about non-normal 2x2 matrices , in spite of the mysterious structure, like Ando-Arveson’s Theorem.

Conclusion:

Classifications of matrices by complete order isomorphisms

- So far, I don't know how to do 3x3 nilpotent matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

- No full description for the 4 x4 diagonal matrices Diag (1, I, -1, -i).

I dare not to touch $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$