Characterization of operator monotone and concave functions in several variables

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Introduction

In this talk, E will denote a Hilbert space.

- $\mathbb{S}(E)$ denote the space of self-adjoint operators
- $\mathbb{S}_n$ is its finite n-by-n dimensional part
- $\mathbb{P} \subseteq \mathbb{S}$ denotes the cone of invertible positive definite and $\hat{\mathbb{P}}$ the cone of positive semi-definite operators
- $\mathbb{P}_n$ and $\hat{\mathbb{P}}_n$ denote the finite dimensional parts

$\mathbb{S}$ and $\mathbb{P}$ are partially ordered cones with the positive definite order:

$$A \leq B \text{ iff } B - A \text{ is positive semidefinite}$$

Free functions

Definition (Free function)

A several variable function $F : D(E) \mapsto \mathbb{S}(E)$ for a domain $D(E) \subseteq \mathbb{S}(E)^k$ defined for all Hilbert spaces E is called a *free* or *noncommutative function* (NC function) if for all E and all $A, B \in D(E) \subseteq \mathbb{S}(E)^k$

(1) $F(U^*A_1U, \ldots, U^*A_kU) = U^*F(A_1, \ldots, A_k)U$ for all unitary $U^{-1} = U^* \in \mathcal{B}(E)$,

(2) $F\left(\begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \ldots, \begin{bmatrix} A_k & 0 \\ 0 & B_k \end{bmatrix}\right) = \begin{bmatrix} F(A_1, \ldots, A_k) & 0 \\ 0 & F(B_1, \ldots, B_k) \end{bmatrix}$.

It follows: the domain $D(E)$ is closed under direct sums and element-wise unitary conjugation, i.e. $D = (D(E))$ is a *free set*.

Operator monotone, concave functions

Definition (Operator monotonicity)

An free function $F : \mathbb{P}^k \mapsto \mathbb{P}$ is operator monotone if for all $X, Y \in \mathbb{P}(E)^k$ s.t. $X \leq Y$, that is $\forall i \in \{1, \ldots, k\} : X_i \leq Y_i$, we have

$$F(X) \leq F(Y).$$

If this property is verified only (hence up to) in dimension n, then F is n-monotone.

Definition (Operator concavity & convexity)

A free function $F : \mathbb{P}^k \mapsto \mathbb{P}$ is operator concave if for all $X, Y \in \mathbb{P}(E)^k$ and $\lambda \in [0, 1]$, we have

$$(1-\lambda)F(A) + \lambda F(B) \leq F((1-\lambda)A + \lambda B)$$

Similarly we define n-concavity.

Loewner's theorem

Theorem (Loewner 1934)

A real function f is operator monotone if and only if

$$f(x) = \alpha + \beta x + \int_0^\infty \frac{\lambda}{\lambda^2+1} - \frac{1}{\lambda+x} d\mu(\lambda),$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ *and* μ *is a unique positive measure on* $[0,\infty)$ *such that* $\int_0^\infty \frac{1}{\lambda^2+1} d\mu(\lambda) < \infty$; *if and only if it has an analytic continuation to the open upper complex half-plane* $\mathbb{H}^+$, *mapping* $\mathbb{H}^+$ *to* $\mathbb{H}^+$.

Some real operator monotone functions on $\mathbb{P}$:

- x^t for $t \in [0,1]$;
- $\log x$;
- $\frac{x-1}{\log x}$.

Operator connections & means

Theorem (Kubo-Ando 1980)

An operator connection $F : \mathbb{P}^2 \mapsto \mathbb{P}$, that is, an operator monotone free function also satisfying $CF(X_1, X_2)C \leq F(CX_1C, CX_2C)$ for all $C^* = C$, can be uniquely represented as

$$F(X) = F(X_1, X_2) = X_1^{1/2} f\left(X_1^{-1/2} X_2 X_1^{-1/2}\right) X_1^{1/2}$$

where $f : (0, \infty) \mapsto (0, \infty)$ is a real operator monotone function.

Some operator connections on $\mathbb{P}^2$:

- Arithmetic mean: $\frac{A+B}{2}$
- Parallel sum: $A : B = \left(A^{-1} + B^{-1}\right)^{-1}$
- Geometric mean: $A\#B = A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^{1/2} A^{1/2}$

Proposition

A concave free function $F : \mathbb{P}^k \mapsto \mathbb{S}$ which is locally bounded from below, is continuous in the norm topology.

Proposition (Hansen type theorem)

Let $F : \mathbb{P}^k \mapsto \mathbb{S}$ be a $2n$-monotone free function. Then F is n-concave, moreover it is norm continuous.

Corollary

An operator monotone free function $F : \mathbb{P}^k \mapsto \mathbb{S}$ is operator concave and norm continuous, moreover it is strong operator continuous on order bounded sets over separable Hilbert spaces E.

The reverse implication is also true if F is bounded from below:

Theorem

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be operator concave (n-concave) free function. Then F is operator monotone (n-monotone).

Supporting linear pencils and hypographs

Definition (Matrix/Freely convex sets of Wittstock)

A graded set $C = (C(E))$, where each $C(E) \subseteq \mathbb{S}(E)^k$, is a bounded open/closed *matrix convex* or *freely convex* set if

(i) each $C(E)$ is open/closed;

(ii) C respects direct sums, i.e. if $(X_1, \ldots, X_k) \in C(N)$ and $(Y_1, \ldots, Y_k) \in C(K)$ and $Z_j := X_j \oplus Y_j$, then $(Z_1, \ldots, Z_k) \in C(N \oplus K)$;

(iii) C respects conjugation with isometries, i.e. if $Y \in C(N)$ and $T : K \mapsto N$ is an isometry, then $T^*YT = (T^*Y_1T, \ldots, T^*Y_kT) \in C(K)$;

(iv) each $C(E)$ is bounded.

The above definition has some equivalent characterizations under slight additional assumptions.

Definition

A graded set $C = (C(E))$, where each $C(E) \subseteq \mathbb{S}(E)^k$, is *closed with respect to reducing subspaces* if for any tuple of operators $(X_1, \ldots, X_k) \in C(E)$ and any corresponding mutually invariant closed subspace $K \subseteq E$, the restricted tuple $(\hat{X}_1, \ldots, \hat{X}_k) \in C(K)$, where each $\hat{X}_i$ is the restriction of X_i to the invariant subspace K for all $1 \leq i \leq k$.

Lemma (Helton, McCullogh 2004)

Suppose that $C = (C(E))$ is a free set, where each $C(E) \subseteq \mathbb{S}(E)^k$, i.e. respects direct sums and unitary conjugation. Then:

(1) *If C is closed with respect to reducing subspaces then C is matrix convex if and only if each $C(E)$ is convex in the usual sense of taking scalar convex combinations.*

(2) *If C is (nonempty and) matrix convex, then $0 = (0, \ldots, 0) \in C(\mathbb{C})$ if and only if C is closed with respect to simultaneous conjugation by contractions.*

Given a set $A \subseteq \mathbb{S}(E)$ we define its *saturation* as

$$\mathrm{sat}(A) := \{X \in \mathbb{S}(E) : \exists Y \in A, Y \geq X\}.$$

Similarly for a graded set $C = (C(E))$, where each $C(E) \subseteq \mathbb{S}(E)$, its *saturation* $\mathrm{sat}(C)$ is the disjoint union of $\mathrm{sat}(C(E))$ for each E.

Definition (Hypographs)

Let $F : \mathbb{P}^k \mapsto \mathbb{S}$ be a free function. Then we define its *hypograph* $\mathrm{hypo}(F)$ as the graded union of the saturation of its image, i.e.

$$\mathrm{hypo}(F) = (\mathrm{hypo}(F)(E)) := (\{(Y, X) \in \mathbb{S}(E) \times \mathbb{P}(E)^k : Y \leq F(X)\}).$$

Characterization of operator concavity

Theorem

Let $F : \mathbb{P}^k \mapsto \mathbb{S}$ be a free function. Then its hypograph $\mathrm{hypo}(F)$ is a matrix convex set if and only if F is operator concave.

Corollary

Let $F : \mathbb{P}^k \mapsto \mathbb{S}$ be a free function. Then its hypograph $\mathrm{hypo}(F)$ is a matrix convex set if and only if F is operator monotone.

Linear pencils

Definition (linear pencil)

A *linear pencil* for $x \in \mathbb{C}^k$ is an expression of the form

$$L_A(x) := A_0 + A_1 x_1 + \cdots + A_k x_k$$

where each $A_i \in \mathbb{S}(K)$ and $\dim(K)$ is the *size* of the pencil L_A.
The pencil is *monic* if $A_0 = I$ and then L_A is a *monic linear pencil*.
We extend the evaluation of L_A from scalars to operators by tensor multiplication. In particular L_A evaluates at a tuple $X \in \mathbb{S}(N)^k$ as

$$L_A(X) := A_0 \otimes I_N + A_1 \otimes X_1 + \cdots + A_k \otimes X_k.$$

We then regard $L_A(X)$ as a self-adjoint element of $\mathbb{S}(K \otimes N)$ and L_A becomes a 'free' function that maps into larger dimensional algebras.

Representation of supporting linear functionals

Suppose $C = (C(E)) \subseteq \mathbb{S}(E)^k$ is a norm closed matrix convex set that is closed with respect to reducing subspaces and $0 \in C(\mathbb{C})$. Then fore each boundary point $A \in C(N)$ where $\dim(N) < \infty$, by the Hahn-Banach theorem there exists a continuous supporting linear functional $\Lambda \in (\mathbb{S}(N)^k)^*$ s.t.

$$\Lambda(C(N)) \leq 1 \text{ and } \Lambda(A) = 1$$

and since $\mathbb{S}(N)^* \simeq \mathbb{S}(N)$ we have that for all $X \in \mathbb{S}(N)^k$

$$\Lambda(X) = \sum_{i=1}^{k} \operatorname{tr}\{B_i X_i\}$$

for some $B_i \in \mathbb{S}(N)$.

Representation of supporting linear functionals

Proposition

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be an operator monotone function and let N be a Hilbert space with $\dim(N) < \infty$. Then for each $A \in \mathbb{P}(N)^k$ and each unit vector $v \in N$ there exists a linear pencil

$$L_{F,A,v}(Y, X) := B(F, A, v)_0 \otimes I - vv^* \otimes Y + \sum_{i=1}^{k} B(F, A, v)_i \otimes (X_i - I)$$

of size $\dim(N)$ which satisfies the following properties:

(1) $B(F, A, v)_i \in \hat{\mathbb{P}}(N)$ *and* $\sum_{i=1}^{k} B(F, A, v)_i \leq B(F, A, v)_0$*;*

(2) *For all* $(Y, X) \in \mathrm{hypo}(F)$ *we have* $L_{F,A,v}(Y, X) \geq 0$*;*

(3) *If* $c_1 I \leq A_i \leq c_2 I$ *for all* $1 \leq i \leq k$ *and some fixed real constants* $c_2 > c_1 > 0$*, then* $\mathrm{tr}\{B(F, A, v)_0\} \leq \frac{F(c_2, \ldots, c_2)}{\min(1, c_1)}$.

Explicit LMI solution formula

Theorem

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be an operator monotone function. Then for each $A \in \mathbb{P}(N)^k$ with $\dim(N) < \infty$ and each unit vector $v \in N$

$$\begin{aligned}
F(A)v &= v^* B_{0,11}(F,A,v)v \otimes Iv + \sum_{i=1}^{k} v^* B_{i,11}(F,A,v)v \otimes (A_i - I)v \\
&\quad - \Bigg\{ (v^* \otimes I) \left[B_{0,12}(F,A,v) \otimes I + \sum_{i=1}^{k} B_{i,12}(F,A,v) \otimes (A_i - I) \right] \\
&\quad \times \left[B_{0,22}(F,A,v) \otimes I + \sum_{i=1}^{k} B_{i,22}(A,v) \otimes (A_i - I) \right]^{-1} \\
&\quad \times \left[B_{0,21}(F,A,v) \otimes I + \sum_{i=1}^{k} B_{i,21}(F,A,v) \otimes (A_i - I) \right] (v \otimes I) \Bigg\} v
\end{aligned}$$

and

$$\Big\{ \Big[\overline{B}_{0,22}(A,v) \otimes I + \sum_{i=1}^{k} B_{i,22}(A,v) \otimes A_i \Big] - \Big[\overline{B}_{0,21}(A,v) \otimes I + \sum_{i=1}^{k} B_{i,21}(A,v) \otimes A_i \Big] \Big\} (v^* \otimes v) = \sum_{j \in \mathcal{I}} \Big[\overline{B}_{0,22}(A,v) \otimes I + \sum_{i=1}^{k} B_{i,22}(A,v) \otimes A_i \Big] (e_j^* \otimes e_j),$$

where $\{e_j\}_{j\in\mathcal{J}}$ is an orthonormal basis of N and

$$\begin{aligned} B_{i,11}(F,A,v) :=& vv^* B_i(F,A,v) vv^*, \\ B_{i,12}(F,A,v) :=& vv^* B_i(F,A,v)(I - vv^*), \\ B_{i,21}(F,A,v) :=& (I - vv^*) B_i(F,A,v) vv^*, \\ B_{i,22}(F,A,v) :=& (I - vv^*) B_i(F,A,v)(I - vv^*) \end{aligned}$$

for all $0 \leq i \leq k$ and $x, y \in \{1, 2\}$.
Moreover if $c_1 I \leq A_i \leq c_2 I$ for all $1 \leq i \leq k$ and some fixed real constants $c_2 > c_1 > 0$, then

$$\operatorname{tr}\{B_0(A, v)\} \leq \frac{F(c_2, \ldots, c_2)}{\min(1, c_1)}.$$

Definition (Natural map)

A graded map $F : \mathbb{S}(K)^k \times K \mapsto K$ defined for all Hilbert space K is called a *natural map* if it preserves direct sums, i.e.

$$F(X \oplus Y, v \oplus w) = F(X, v) \oplus F(Y, w)$$

for $X \in \mathbb{S}(K_1)^k$, $v \in K_1$ and $Y \in \mathbb{S}(K_2)^k$, $w \in K_2$.

For a free function $F : \mathbb{S}^k \mapsto \mathbb{S}$ we define the natural map $\overline{F} : \mathbb{S}(K)^k \times K \mapsto K$ for any K by

$$\overline{F}(X, v) := F(X)v$$

for $X \in \mathbb{S}(K)^k$ and $v \in K$.

The function below is free, hence induces a natural map:

$$\begin{aligned} F(X) :=& v^* B_{0,11} v \otimes I + \sum_{i=1}^{k} v^* B_{i,11} v \otimes X_i \\ & - (v^* \otimes I) \left[B_{0,12} \otimes I + \sum_{i=1}^{k} B_{i,12} \otimes X_i \right] \\ & \times \left[B_{0,22} \otimes I + \sum_{i=1}^{k} B_{i,22} \otimes X_i \right]^{-1} \\ & \times \left[B_{0,21} \otimes I + \sum_{i=1}^{k} B_{i,21} \otimes X_i \right] (v \otimes I). \end{aligned}$$

Let $S(E) := \{v \in E : \|v\| = 1\}$ denote the unit sphere of the Hilbert space E. For fixed real constants $c_2 > c_1 > 0$, let

$$\mathbb{P}_{c_1,c_2}(E) := \{X \in \mathbb{P}(E) : c_1 I \leq X \leq c_2 I\},$$
$$\Omega_{c_1,c_2} := \mathbb{P}_{c_1,c_2}(E)^k \times S(E)$$

and let

$$\mathcal{H} := \bigoplus_{\dim(E)<\infty} \bigoplus_{\omega \in \Omega_{c_1,c_2}} E.$$

We equip $\mathcal{H}$ with the inner product

$$x^*y := \sum_{\dim(E)<\infty} \sum_{\omega \in \Omega_{c_1,c_2}} x(\omega)^* y(\omega).$$

Let $\mathcal{B}^+(\mathcal{H})^*$ denote the state space of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}^+(\mathcal{H})_*$ is the normal part.

Definition

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be an operator monotone function. Now let

$$\begin{aligned}\Psi_F(X) := & B_{0,11} \otimes I + \sum_{i=1}^{k} B_{i,11} \otimes (X_i - I) \\ & - \left[B_{0,12} \otimes I + \sum_{i=1}^{k} B_{i,12} \otimes (X_i - I) \right] \\ & \times \left[B_{0,22} \otimes I + \sum_{i=1}^{k} B_{i,22} \otimes (X_i - I) \right]^{-1} \\ & \times \left[B_{0,21} \otimes I + \sum_{i=1}^{k} B_{i,21} \otimes (X_i - I) \right]\end{aligned}$$

where

$$B_{0,xy} := \bigoplus_{\dim(E)<\infty} \bigoplus_{(A,v)\in\Omega_{c_1,c_2}} B_{0,xy}(F,A,v),$$

$$B_{i,xy} := \bigoplus_{\dim(E)<\infty} \bigoplus_{(A,v)\in\Omega_{c_1,c_2}} B_{i,xy}(F,A,v)$$

for $1 \leq i \leq k$ and $x, y \in \{1, 2\}$.

Lemma

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be an operator monotone function and let $\dim(E) < \infty$. Let $A_j \in \mathbb{P}_{c_1,c_2}(E)^k$ and $v_j \in S(E)$ for $j \in \mathcal{J}$ for some finite index set $\mathcal{J}$. Then there exists a $w \in S(\mathcal{H})$ such that

$$F(A_j)v_j = (w^* \otimes I)\Psi_F(A_j)(w \otimes I)v_j$$

for all $j \in \mathcal{J}$.

Theorem (Multivariable Loewner's theorem)

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be an operator monotone function. Then there exists a state $\omega \in \mathcal{B}_1^+(\mathcal{H})^*$ such that for all $\dim(E) < \infty$ and $X \in \mathbb{P}(E)^k$ we have

$$
\begin{aligned}
F(X) =& (\omega \otimes I)(\Psi_F(X)) = \omega(B_{0,11}) \otimes I + \sum_{i=1}^{k} \omega(B_{i,11}) \otimes (X_i - I) \\
& - (\omega \otimes I) \left\{ \left[B_{0,12} \otimes I + \sum_{i=1}^{k} B_{i,12} \otimes (X_i - I) \right] \right. \\
& \times \left[B_{0,22} \otimes I + \sum_{i=1}^{k} B_{i,22} \otimes (X_i - I) \right]^{-1} \\
& \left. \times \left[B_{0,21} \otimes I + \sum_{i=1}^{k} B_{i,21} \otimes (X_i - I) \right] \right\}.
\end{aligned}
$$

The upper operator half-space $\Pi(E)$ consists of $X \in \mathcal{B}(E)$ s.t. $\Im X := \frac{X - X^*}{2i} > 0$.

Theorem (Multivariable Loewner's theorem cont.)

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be a free function. Then the following are equivalent

(1) *F is operator monotone;*

(2) *F is operator concave;*

(3) *F is a conditional expectation of the Schur complement of a linear pencil $L_B(X) := B_0 \otimes I + \sum_{i=1}^{k} B_i \otimes (X_i - I)$ over some auxiliary Hilbert space $\mathcal{H}$ with $B_i \in \hat{\mathbb{P}}(\mathcal{H})$, $B_0 \geq \sum_{i=1}^{k} B_i$;*

(4) *F admits a free analytic continuation to the upper operator poly-halfspace $\Pi(E)^k$, mapping $\Pi(E)^k$ to $\Pi(E)$ for all E.*

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Thank you for your kind attention!