Algebraic curves associated to some matrices

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Outline

1. Introduction

2. Singular points

3. General hyperbolic forms
The complex projective algebraic curve $C_F$ of an arbitrary ternary homogeneous polynomial $F(t, x, y)$ is defined by

$$C_F = \{(t, x, y) \in \mathbb{CP}^2 : F(t, x, y) = 0\}.$$ 

where $\mathbb{CP}^2$ denotes the complex projective plane, that is, the quotient space $\mathbb{C}^3 \setminus \{(0, 0, 0)\}/\sim$ with respect to the relation $(t_1, x_1, y_1) \sim (t_2, x_2, y_2)$ if $(t_2, x_2, y_2) = c(t_1, x_1, y_1)$ for a non-zero complex number $c$.

The dual curve of $C_F$ is defined by

$$\Gamma_F = \{(T_0, X_0, Y_0) \in \mathbb{CP}^2 : T_0 t + X_0 x + Y_0 y = 0 \text{ is a tangent line of } C_F\}.$$
Let $A$ be an $n \times n$ matrix.

Denote $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/(2i)$.

A complex projective algebraic curve associated to $A$ is defined by the associated homogeneous polynomial

$$F_A(t, x, y) = \det \left( tl_n + x \Re(A) + y \Im(A) \right).$$

Numerical range

$$W(A) = \{ x^*A x; x \in \mathbb{C}^n, \|x\| = 1 \}.$$
1. Introduction

\[ A = \begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}. \]

\[ F_A(1, x, y) = 0 \quad \text{Dual curve (flat portion)} \]
For arbitrary complex numbers $a_1, a_2, \ldots, a_n$, we consider an $n \times n$ cyclic weighted shift matrix $S(a_1, a_2, \ldots, a_n)$ defined as

$$S = S(a_1, a_2, \ldots, a_n) = \begin{pmatrix} 0 & a_1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & a_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \vdots & \vdots & \vdots & a_{n-1} \\ a_n & 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix}.$$
1. Introduction

- M. C. Tsai, P. Y. Wu, 2011
  M. C. Tsai, H. L. Gau, H. C. Wang, 2014

Let $S$ be an $n \times n$ cyclic weighted shift matrix. Then the boundary of $W(S)$ has a flat portion if and only if the weights are nonzero and the numerical ranges of three $(n - 1) \times (n - 1)$ principal submatrices of $S$ are equal.
A cyclic weighted shift matrix $S(a_1, a_2, \ldots, a_n)$ is unitarily irreducible if and only if its weights $a_1, a_2, \ldots, a_n$ are non-periodic.
1. Introduction

I will focus on the following topics

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2. Find the number of the singular points of $F_S(t, x, y) = 0$. 

1. Introduction

I will focus on the following topics

1. Investigate the types of the singular points of the curve $F_S(t, x, y) = 0$.
2. Find the number of the singular points of $F_S(t, x, y) = 0$.
3. Extend to general hyperbolic forms.
2. Singular points

A point \((t_0, x_0, y_0)\) of the curve \(F(t, x, y) = 0\) is called a singular point if \(F(t_0, x_0, y_0) = 0\) and

\[
\frac{\partial}{\partial t} F(t_0, x_0, y_0) = \frac{\partial}{\partial x} F(t_0, x_0, y_0) = \frac{\partial}{\partial y} F(t_0, x_0, y_0) = 0.
\]
2. Singular points

A point \((t_0, x_0, y_0)\) of the curve \(F(t, x, y) = 0\) is called a **singular point** if \(F(t_0, x_0, y_0) = 0\) and

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\]

- If the boundary of \(W(A)\) has a flat portion on the line \(a_0 + a_1 x + a_2 y = 0\) then the point \((a_0, a_1, a_2)\) is a singular point of the curve \(F_A(t, x, y) = 0\).
2. Singular points

A singular point \((t_0, x_0, y_0)\) of the curve is called a **double point** if at least one of the second derivatives

\[
\frac{\partial^2}{\partial t^2} F(t_0, x_0, y_0), \quad \frac{\partial^2}{\partial x^2} F(t_0, x_0, y_0), \quad \frac{\partial^2}{\partial y^2} F(t_0, x_0, y_0),
\]

\[
\frac{\partial^2}{\partial t \partial x} F(t_0, x_0, y_0), \quad \frac{\partial^2}{\partial t \partial y} F(t_0, x_0, y_0), \quad \frac{\partial^2}{\partial x \partial y} F(t_0, x_0, y_0)
\]

does not vanish.
2. Singular points

A singular point \((t_0, x_0, y_0)\) of the curve is called a **double point** if at least one of the second derivatives

\[
\frac{\partial^2}{\partial t^2} F(t_0, x_0, y_0), \quad \frac{\partial^2}{\partial x^2} F(t_0, x_0, y_0), \quad \frac{\partial^2}{\partial y^2} F(t_0, x_0, y_0),
\]

\[
\frac{\partial^2}{\partial t \partial x} F(t_0, x_0, y_0), \quad \frac{\partial^2}{\partial t \partial y} F(t_0, x_0, y_0), \quad \frac{\partial^2}{\partial x \partial y} F(t_0, x_0, y_0)
\]

does not vanish.

A double point of a curve is called a **node** (or an ordinary double point) if there are two distinct tangents at the point.
2. Singular points

Nakazato-Chien, 2012

The shapes of the numerical ranges of $4 \times 4$ matrices $A$ are determined according to the classification of singular points of $F_A(t, x, y) = 0$

For an irreducible ternary form, there are 21 types according to the number of its singular points and their forms

- 1 triple point
- 3 nodes
- 2 nodes
- 1 node
- 1 node and 1 tacnode
- 1 osnode
- 1 tacnode
- 2 cusps
- 2 cusps and 1 node
- no singular points
2. Singular points

- Triple point
- 3 nodes
- Tacnode
- One flat
- 3 flats
- No flat
2. Singular points

Nakazato-Chien, 2013

**Theorem 2.1** Let \( S(a_1, a_2, \ldots, a_n) \) be a cyclic weighted shift matrix with non-zero weights. Then all singular points of the complex projective curve \( F_S(t, x, y) = 0 \) are nodes.
Nakazato-Chien, 2013

**Theorem 2.1** Let $S(a_1, a_2, \ldots, a_n)$ be a cyclic weighted shift matrix with non-zero weights. Then all singular points of the complex projective curve $F_S(t, x, y) = 0$ are nodes.

- The singular points of the curve $F_S(t, x, y) = 0$ belong to the real projective plane.
- A real singular point of $F_S(t, x, y) = 0$ on the line $t = 0$ can be assumed $(0, 1, 0)$, the tangents are $t = \pm ay$.
- A real singular point $(1, x_0, y_0)$ of $F_S(t, x, y) = 0$ can be assumed $(1, 1, 0)$, the tangents are $\pm by + x - 1 = 0$. 
Nakazato-Chien, 2013

**Theorem 2.2** Let $S(a_1, a_2, \ldots, a_n)$ be a cyclic weighted shift matrix with non-zero weights. Then the number of singular points of the curve $F_S(t, x, y) = 0$ is at most $n(n - 1)/2$. 

• $F_S(t, x, y)$ has no repeated factors.
Nakazato-Chien, 2013

**Theorem 2.2** Let $S(a_1, a_2, \ldots, a_n)$ be a cyclic weighted shift matrix with non-zero weights. Then the number of singular points of the curve $F_S(t, x, y) = 0$ is at most $n(n - 1)/2$.

- $F_S(t, x, y)$ has no repeated factors.
- Then, by an algebraic curve theorem, the number of double points $\leq n(n - 1)/2$. 
2. Singular points

How many singular points of $F_S(t, x, y) = 0$?
1. Singular points

$n(n - 1)/2$

The upper bound $n(n - 1)/2$ is attained for any $n \geq 3$ by the canonical cyclic shift matrix $S(1, 1, \ldots, 1)$.

$$F_S(t, x, y) = \prod_{k=0}^{n-1} \left( t + \cos(2k\pi/n)x + \sin(2k\pi/n)y \right).$$

The singular points of the curve $F_S(t, x, y) = 0$ are given by

$$\{(t, x, y) = (1, -\frac{\cos((k + \ell)\pi/n)}{\cos((k - \ell)\pi/n)}, -\frac{\sin((k + \ell)\pi/n)}{\cos((k - \ell)\pi/n)} : 0 \leq \ell < k \leq n}\}. $$
2. Singular points

① \( n(n - 1)/2 \)  ② \( n(n - 2)/2 \)

Nakazato-Chien, 2013

**Theorem 2.3** Let \( n \geq 4 \) be an even number and \( S(a_1, a_2, \ldots, a_n) \) be a cyclic weighted shift matrix with non-zero weights. Then the number of singular points of the curve \( F_S(t, x, y) = 0 \) is \( n(n - 2)/2 \) if and only if the form \( F_S(t, x, y) \) is the product of \( n/2 \) quadratic forms.

In this case, \( S(a_1, a_2, \ldots, a_n) \) is *unitarily reducible* and the weights are 2-periodic, that is, \( a_{2j-1} = \alpha, a_{2j} = \beta, j = 1, 2, \ldots, n/2. \)
2. Singular points

① $n(n - 1)/2$  ② $n(n - 2)/2$  ③ $n(n - 3)/2$

Nakazato-Chien, 2013

**Theorem 2.4** Let $S = S(a_1, a_2, \ldots, a_n)$, $n \geq 3$, be a unitarily irreducible cyclic weighted shift matrix with non-zero weights. Then the number of singular points of the associative curve $F_S(t, x, y) = 0$ is at most $n(n - 3)/2$. 

2. Singular points

\[ n(n - 1)/2 \quad 2 \quad n(n - 2)/2 \quad 3 \quad n(n - 3)/2 \]

Nakazato-Chien, 2013

**Theorem 2.4** Let \( S = S(a_1, a_2, \ldots, a_n), \ n \geq 3, \) be a unitarily irreducible cyclic weighted shift matrix with non-zero weights. Then the number of singular points of the associative curve \( F_S(t, x, y) = 0 \) is at most \( n(n - 3)/2. \)

- Suppose \((1, x_0, y_0)\) is a singular point of \( F_S(t, x, y) = 0. \) By the symmetry,
  \[
  F_S(t, \cos(2\pi/n)x - \sin(2\pi/n)y, \sin(2\pi/n)x + \cos(2\pi/n)y) = F_S(t, x, y),
  \]
the \( n \) points
  \[
  (1, x_0 \cos(2k\pi/n) - y_0 \sin(2k\pi/n), x_0 \sin(2k\pi/n) + y_0 \cos(2k\pi/n))
  \]
\( k = 0, 1, 2, \ldots, n - 1 \) are singular points of \( F_S(t, x, y) = 0. \)
2. Singular points

1. \( n(n - 1)/2 \)  
2. \( n(n - 2)/2 \)  
3. \( n(n - 3)/2 \)

- \( n \) is an odd number: Then the curve \( F_S(t, x, y) = 0 \) has no singular points on the line \( t = 0 \). Hence the possible numbers of singular points of \( F_S(t, x, y) = 0 \) are

\[ \{ n(n - 1)/2, n(n - 3)/2, n(n - 5)/2, \ldots, n, 0 \}. \]

- \( n \) is an even number \( \geq 4 \): If \((0, x_0, y_0)\) is a singular point of the curve \( F_S(t, x, y) = 0 \), where \((x_0, y_0) \in \mathbb{R}^2\), then the \( n/2 \) points

\[ (0, x_0 \cos(2k\pi/n) - y_0 \sin(2k\pi/n), x_0 \sin(2k\pi/n) + y_0 \cos(2k\pi/n)) \]

\( k = 0, 1, 2, \ldots, (n/2) - 1 \), are singular points of the curve \( F_S(t, x, y) = 0 \). Hence, the number of singular points of the curve \( F_S(t, x, y) = 0 \) is one of the following:

\[ \{ n(n - 1)/2, n(n - 2)/2, n(n - 3)/2, \ldots, n, n/2, 0 \}. \]
2. Singular points

Conjecture

The upper bound $n(n - 3)/2$ is sharp for the numbers of the singular points of $F_S(t, x, y) = 0$ associated with $n \times n$ unitarily irreducible cyclic weighted shift matrices.
Partial Answer: The conjecture is true for $n = 4, 5, 6, 7$.

For examples:

$n = 4$. $a_1a_3 = a_2a_4$ and $(a_1 - a_3)^2 + (a_2 - a_4)^2 > 0$, the curve $F_S(t, x, y) = 0$ has $n(n - 3)/2 = 2$ nodes at $(t, x, y) = (0, 1, 0), (0, 0, 1)$. 
Partial Answer: The conjecture is true for \( n = 4, 5, 6, 7 \).

For examples:

\( n = 4 \). \( a_1 a_3 = a_2 a_4 \) and \( (a_1 - a_3)^2 + (a_2 - a_4)^2 > 0 \), the curve \( F_S(t, x, y) = 0 \) has \( n(n - 3)/2 = 2 \) nodes at \( (t, x, y) = (0, 1, 0), (0, 0, 1) \).

\( n = 6 \). \( a_1 = a_6 = 2\sqrt{2}, a_2 = a_5 = 2 \) and \( a_3 = a_4 = \sqrt{8/3} \), the curve \( F_S(t, x, y) = 0 \) has \( n(n - 3)/2 = 9 \) nodes at \( (t, x, y) = (1, \cos(k\pi/3), \sin(k\pi/3)), \ k = 0, 1, \ldots, 5, \) and \( (t, x, y) = (0, 0, 1), (0, \cos(\pi/6), \sin(\pi/6)), (0, \cos(\pi/6), \sin(\pi/6)) \).
3. General hyperbolic forms

The real ternary form $F_S(t, x, y)$ of a cyclic weighted shift matrix $S = S(a_1, a_2, \ldots, a_n)$ with non-zero weights satisfies the following conditions:

(i) $F_S(t, x, y)$ is hyperbolic w.r.t. $(1, 0, 0)$ and $F(1, 0, 0) = 1$.

(ii) $F_S(t, x, y)$ is weakly circular symmetric:

$$F_S(t, \cos(2\pi/n)x - \sin(2\pi/n)y, \sin(2\pi/n)x + \cos(2\pi/n)y) = F_S(t, x, y).$$

(iii) $F_S(t, x, -y) = F_S(t, x, y)$.

(iv) $F_S(t, -1, -i) = t^n - a$ for some nonzero real number $a$. 
3. General hyperbolic forms

**Theorem 3.1** Let $F(t, x, y)$ be a real ternary form of degree $n$ satisfying conditions (i)-(iv). Then all singular points of the complex projective curve $F(t, x, y) = 0$ are nodes.
3. General hyperbolic forms

**Theorem 3.1** Let $F(t, x, y)$ be a real ternary form of degree $n$ satisfying conditions $(i)$-$(iv)$. Then all singular points of the complex projective curve $F(t, x, y) = 0$ are nodes.

**Theorem 3.2** Let $F(t, x, y)$ be a real ternary form of degree $n$ satisfying conditions $(i)$-$(iv)$. Then $F(t, x, y)$ has no repeated factors, and the number of singular points of the complex projective curve $F(t, x, y) = 0$ is at most $n(n - 1)/2$. 
References


Thank you for your attention!