Applications of matrix theory to quantum coherence

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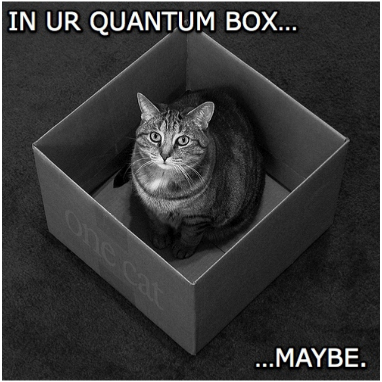
Outline

Classical versus Quantum—What's the Difference?

In short, entanglement and superpositions.



Superpositions?

When multiple quantum systems interact with each other, the resource of interest is typically *entanglement*.

When there is no interaction between different systems, the resource of interest is instead *coherence*, or the amount that a state is in a superposition of a given set of mutually orthogonal states.

Superpositions are essentially linear combinations of basis states (with the additional property that the coefficients $\alpha_i \in \mathbb{C}$ in the linear combination satisfy $\sum_i |\alpha_i|^2 = 1$)

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Also, Schrödinger's cat.

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Physical Motivation

One of the major goals in quantum information theory is to find effective ways of quantifying the amount of "quantumness" within a given system—that is, how much the system differs from any possible classical mechanical description of it.

Some Notation

- Restrict attention to unit vectors
- Unit vectors $\mathbf{x} \in \mathbb{C}^n$ represent quantum information in our quantum system $\mathbb{C}^n$ and are called quantum *states*
- States are either **pure** (represented by a unit vector $\mathbf{x} \in \mathbb{C}^n$ or, equivalently, its outer product $\mathbf{x}\mathbf{x}^*$, which is a rank-one projection) or **mixed** (represented by a density matrix $\rho = \sum_i p_i \mathbf{x}\mathbf{x}^*$, where $\{p_i\}$ forms a probability distribution).
- Density matrices (including outer products associated to pure states) are precisely the trace-one, positive (semi-definite) matrices.

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Mathematical Framework

Fix an orthonormal basis $\{\mathbf{v_i}\}_{i=1}^{n}$ of $\mathbb{C}^n$. An *incoherent state* is any density matrix that is diagonal in this basis—that is, any matrix of the form $\sum_{i=1}^{n} p_i \mathbf{v_i}\mathbf{v_i}^*$ where $0 \leq p_i \leq 1$ for all i and $\sum_{i=1}^{n} p_i = 1$.

Note: Incoherent = no coherence. That is, diagonal density matrices can be seen as "classical" in that there are no superpositions.

We consider the distance of a given state to the closest incoherent state via a *coherence measure*:

$$C(\rho) = \min_{\delta \in \mathcal{I}} d(\rho - \delta)$$

where $\mathcal{I}$ is the set of all incoherent states in the quantum system and d is a distance measure defined on the system.

—this is *highly* basis-dependent!

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Measuring Coherence

The two most widely-known coherence measures are:

- the ℓ_1-*norm of coherence*:

$$C_{\ell_1}(\rho) := \min_{\delta\in\mathcal{I}} \|\rho-\delta\|_{\ell_1} = \min_{\delta\in\mathcal{I}} \sum_{i,j=1}^{n} |\rho-\delta|_{ij} = \sum_{i\neq j} |\rho_{ij}|,$$

- the *relative entropy of coherence*:

$$C_r(\rho) := S(\rho_{\text{diag}}) - S(\rho),$$

where $S(\rho) = -\mathrm{tr}(\rho\log_2\rho)$ is the von Neumann entropy of the state ρ and ρ_{diag} is the state obtained from ρ by deleting all off-diagonal entries.

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Other coherence measures that have recently been proposed:

- the *trace distance of coherence*:

$$C_{\mathrm{Tr}}(\rho) := \min_{\delta\in\mathcal{I}} \|\rho-\delta\|_{\mathrm{Tr}} = \min_{\delta\in\mathcal{I}} \sum_{i=1}^{n} |\lambda_i(\rho-\delta)|,$$

where $\lambda_i(\rho-\delta)$ are the eigenvalues of the matrix $\rho-\delta$ and $\|\cdot\|_{\mathrm{Tr}}$ is the trace norm

- the *robustness of coherence*:

$$C_R(\rho) := \min_{\tau} \left\{ s \geq 0 \,\middle|\, \frac{\rho + s\tau}{1+s} \in \mathcal{I} \right\},$$

where the minimum is over all density matrices τ.
Note $C_R(\mathbf{xx}^*) = C_{\ell_1}(\mathbf{xx}^*)$ (Napoli et al, 2016).

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How good are these measures?

The defining properties of a *proper* coherence measure have recently been identified (Baumgratz-Cramer-Plenio, 2014); for example, a state ρ should have zero coherence under the proposed measure if and only if ρ is incoherent, and the proposed measure should be convex: $\sum p_n C(\rho_n) \geq C(\sum p_n \rho_n)$ for any set of states $\{\rho_n\}$ and any $p_n \geq 0$ with $\sum p_n = 1$.

The ℓ_1-norm of coherence, relative entropy of coherence, and robustness of coherence have all been shown to be proper coherence measures, and it has been shown that the trace distance of coherence is a proper measure of coherence when restricted to qubit states or X states (though the general case remains open).

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2×2 setting

If ρ is 2×2 and $D \in \mathcal{I}$ minimizes $C_{tr}(\rho)$, then $D = \rho_{\text{diag}}$ and hence $C_{tr}(\rho) = C_{\ell_1}(\rho)$ (Rana-Parashar-Lewenstein, 2016).

In their words, "finding an analytic form even for pure qutrits is almost intractable". Evidence was given to suggest that a simple closed-form formula might not exist.

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The trace distance of coherence of a pure state

We wish to characterize $C_{\mathrm{Tr}}(\mathbf{x}\mathbf{x}^*)$, where $\mathbf{x} \in \mathbb{C}^n$ is an arbitrary pure state (unit vector).

We give an "almost formula" for the trace distance of coherence of a pure state: we show that it is given by one of n different formulas (depending on the state), and which formula is the correct one can be determined simply by checking $\log_2(n)$ inequalities.

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The trace distance of coherence of a pure state

Note that there is a diagonal unitary U and a permutation matrix P such that $PU\mathbf{x}$ is a unit vector having non-negative entries $x_1 \geq \cdots \geq x_n \geq 0$ in descending order. We then have

$$\|\mathbf{x}\mathbf{x}^* - \delta\| = \|PU(\mathbf{x}\mathbf{x}^* - \delta)U^*P^t\|$$

for any $\delta \in \mathcal{I}$. So, we may replace $\mathbf{x}$ by $PU\mathbf{x}$ without loss of generality.

The algorithm

Suppose $\mathbf{x} = (x_1, \ldots, x_n)^t$ is a unit vector with entries $x_1 \geq \cdots \geq x_n \geq 0$. Let $s_\ell = \sum_{j=1}^{\ell} x_j$, $m_\ell = \sum_{j=\ell+1}^{n} x_j^2$, and $p_\ell = s_\ell^2 - 1 - \ell m_\ell$ for $\ell \in \{1, \ldots, n\}$. There is a maximum integer $k \in \{1, \ldots, n\}$ satisfying

$$x_k > q_k := \frac{1}{2ks_k}\left(p_k + \sqrt{p_k^2 + 4km_k s_k^2}\right). \tag{1}$$

The unique best approximation of $\mathbf{x}\mathbf{x}^*$ in $\mathcal{I}$ with respect to the trace norm (and the operator norm) is $D = \mathrm{diag}(d_1, \ldots, d_k, 0, \ldots, 0) \in \mathcal{I}$ with

$$d_j = \frac{x_j - q_k}{s_k - kq_k} \quad \text{for} \quad 1 \leq j \leq k.$$

Furthermore,

$$\begin{aligned} C_{\mathrm{Tr}}(\mathbf{x}\mathbf{x}^*) = \|\mathbf{x}\mathbf{x}^* - D\|_{\mathrm{Tr}} &= 2(q_k s_k + m_k), \quad \text{and} \\ \|\mathbf{x}\mathbf{x}^* - D\| &= q_k s_k + m_k. \end{aligned}$$

Although it might seem somewhat time-consuming at first to find the value of k described by the theorem, the proof of the theorem shows that if $q_k < x_k$ then $q_j < x_j$ for all $j < k$. Thus we can search for k via binary search, which requires only $\log_2(n)$ steps, rather than searching through all n possible values of k. MATLAB code that implements this algorithm is able to compute $C_{\mathrm{Tr}}(\mathbf{x}\mathbf{x}^*)$ for pure states $\mathbf{x} \in \mathbb{C}^{1,000,000}$ in under one second on a standard laptop computer.

A pure state $\mathbf{x} \in \mathbb{C}^n$ is *maximally coherent* if all of its entries have equal absolute value: $|x_1| = \cdots = |x_n| = 1/\sqrt{n}$.

Recently it has been suggested that the maximum value of a proper measure of coherence should be attained exactly by the maximally coherent states. This property is known to hold for the relative entropy of coherence, the ℓ_1-norm of coherence, and the robustness of coherence.

Theorem

For all (potentially mixed) states ρ, we have $C_{\mathrm{Tr}}(\rho) \leq 2 - 2/n$. Furthermore, equality holds if and only if $\rho = \mathbf{x}\mathbf{x}^*$, where $\mathbf{x}$ is a maximally coherent state.

This theorem provides further evidence that the trace norm is indeed a proper measure of coherence.

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Relationship between the ℓ_1-norm of coherence and the relative entropy of coherence

Conjecture in

Trace distance measure of coherence

Swapan Rana,[1,*] Preeti Parashar,[2] and Maciej Lewenstein[1,3]

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[2]*Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 BT Road, Kolkata, India*
[3]*ICREA – Institució Catalana de Recerca i Estudis Avançats, 08010 Barcelona, Spain*

(Dated: November 6, 2015)

Theorem in

Quantifying the coherence of pure quantum states

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[4]Department of Mathematics & Computer Science, Brandon University, Brandon, MB, Canada R7A 6A9

(Dated: January 23, 2016)

But also in

Trace-distance measure of coherence

Swapan Rana,[1,*] Preeti Parashar,[2] and Maciej Lewenstein[1,3]

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[2]Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 BT Road, Kolkata, India

[3]ICREA – Institució Catalana de Recerca i Estudis Avançats, 08010 Barcelona, Spain

(Dated: January 22, 2016)

Theorem (Rana-Parashar-Lewenstein, 2016)

For every pure state $\mathbf{x}$*,*

$$C_{\ell_1}(\mathbf{xx}^*) \geq \max\{C_r(\mathbf{xx}^*), 2^{C_r(\mathbf{xx}^*)} - 1\}.$$

The ℓ_1-norm coherence of a pure state is never smaller than its relative entropy of coherence.

Does this hold for mixed states as well?

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What have we done?

- We derived an explicit expression for the trace distance of coherence of a pure state as well as the closest incoherent state to a given pure state with respect to the trace distance.
- We showed that the trace distance of coherence of an arbitrary (pure or mixed) state is maximized iff the state under consideration is a maximally coherent state.
- We gave a more explicit proof to the conjecture relating the ℓ_1-norm of coherence and the relative entropy of coherence

Future Work

- One natural question that arises from this work is whether or not our algorithm can be used to show that the trace distance of coherence is strongly monotonic under incoherent quantum channels (another property of a proper coherence measure), at least when it is restricted to pure states.
- Prove

$$C_{\ell_1}(\rho) \geq \max\{C_r(\rho), 2^{C_r(\rho)} - 1\}$$

for all states ρ.