Primitive symmetric matrices and their preservers

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1. Definitions

A *semiring* is a system, $(\mathbb{S},+,\times)$, where $\mathbb{S}$ is a nonempty set, $(\mathbb{S},+)$ is an Abelian monoid (identity 0), $(\mathbb{S},\times)$ is a monoid (identity 1), $\times$ distributes over $+$, and $0\times s = s\times 0 = 0$ for all $s\in\mathbb{S}$.

The *binary Boolean semiring* $\mathbb{B}=(\{0,1\},+,\times)$. Note that $\mathbb{B}$ has arithmetic the same as real arithmetic except that $1+1 = 1$.

Let $M_n(\mathbb{S})$ denote the set of all $n\times n$ matrices with entries in the semiring $\mathbb{S}$, $S_n(\mathbb{B})$, the set of all $n\times n$ symmetric Boolean matrices, including $S_n^{(0)}(\mathbb{B})$, the set of matrices in $S_n(\mathbb{B})$ with all diagonal entries zero.

We let J_n denote the $n \times n$ matrix of all ones, O_n the $n \times n$ matrix of all zeros, and I_n the $n \times n$ identity matrix and K_n = $J_n - I_n$. The subscripts are suppressed unless there is possibility of confusion and write J, O, I, K respectively. The matrix $E_{i,j}$ in $M_n(\mathbb{S})$ is the $n \times n$ matrix with exactly one nonzero entry, that being a one in the (i,j) position and is called a *cell*. We call $E_{i,i}$ a *diagonal cell*. Let D_n denote the set of all diagonal matrices. The matrix $D_{i,j}$ for $i \neq j$, is the $n \times n$ matrix with $D_{i,j} = E_{i,j} + E_{j,i}$. The matrix $D_{i,j}$ is called a *digon*.

Let $A, B \in M_n(\mathbb{S})$. We say that *A dominates B* (written $B \sqsubseteq A$) if $a_{i,j} = 0$ implies $b_{i,j} = 0$ for all i, j. If B $\sqsubseteq$ A we write $A \setminus B = C$ to denote the matrix with $c_{i,j} = a_{i,j}$ if $b_{i,j} = 0$ and $c_{i,j} = 0$ if $b_{i,j} \neq 0$.

Let $A \circ B$ denote the *Hadamard product* of A and B, that is $A \circ B$ = $[a_{i,j} b_{i,j}]$.

Note that if $X \in M_n(\mathbb{S})$ then $X \circ K \in M_n^{(0)}(\mathbb{S})$ where $M_n^{(0)}(\mathbb{S})$ denotes the set of matrices in $M_n(\mathbb{S})$ with all diagonal entries zero.

A *line matrix* is a matrix of the form $R_i = \sum_{j=1}^{n} E_{i,j}$ or $C_j = \sum_{i=1}^{n} E_{i,j}$.

The *double star* centered on k is the matrix in $\boldsymbol{S}_n(\mathbb{S})$ of the form $D_k = R_k + C_k$, $k = 1,2,\ldots,n$.

Let $\boldsymbol{N}$ be a subsemimodule of $\boldsymbol{M}_n(\mathbb{B})$. A *base element* of $\boldsymbol{N}$ is an element E of $\boldsymbol{N}$ such that E is not an algebraic sum of elements of $\boldsymbol{N} \setminus \{E\}$.

Example 1.1 The following table gives the structure of the base elements of several semimodules, of interest.

Semimodule	Base elements
$M_n(\mathbb{S})$	$E_{i,j}$
$S_n(\mathbb{S})$	$E_{i,i}$ or $D_{i,j}$, $i \neq j$
$S_n^{(0)}(\mathbb{S})$	$D_{i,j}$, $i \neq j$

Let L be a subset of semimodule N. We say that L *separates base elements* if, given any two distinct base elements, E and F, there is some $G \in N$ such that $G+E \in L$ and $G+F \notin L$. In this case we say that L *separates* E from F.

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An *upper ideal* of matrices in $\boldsymbol{N}$ is a set $\boldsymbol{U}$ such that if $A\in\boldsymbol{U}$ and B is any matrix in $\boldsymbol{N}$, then $A+B\in\boldsymbol{U}$.

Let $A\in\boldsymbol{M}_n(\mathbb{B})$. The *(Boolean) rank* of A is the least integer k such that there are an $n\times k$ Boolean matrix B and a $k\times n$ Boolean matrix C with $A=BC$.

Let T: $\boldsymbol{M}_n(\mathbb{S})\rightarrow \boldsymbol{M}_n(\mathbb{S})$ be an operator. Then T is called *linear operator* if $T(\alpha A+\beta B)=\alpha T(A)+\beta T(B)$ for all $\alpha,\beta\in\mathbb{S}$ and all $A,B\in\boldsymbol{M}_n(\mathbb{S})$.

Definition 1.2. A linear operator T: $\boldsymbol{M}_n(\mathbb{S})\rightarrow \boldsymbol{M}_n(\mathbb{S})$ is called a *(P,Q)-operator* if there exist permutation matrices P and Q, such that

$$T(X)=PXQ \quad \text{or} \quad T(X)=PX^tQ \text{ for all } X\in\boldsymbol{M}_n(\mathbb{S}).$$

Definition 1.3. Let Φ be a subset of $M_n(\mathbb{S})$.

(1) T *preserves* Φ if $X \in \Phi$ implies $T(X) \in \Phi$.

(2) T *strongly preserves* Φ if, $X \in \Phi$ if and only if $T(X) \in \Phi$.

Lemma 1.4. Let $T: \mathbf{N} \to \mathbf{N}$ be an idempotent linear operator and $\mathbf{U}$ an upper ideal in $\mathbf{N}$. If T strongly preserves $\mathbf{U}$ and $\mathbf{U}$ separates base element E from base element F then $E \not\sqsubseteq T(F)$.

Proof. Suppose $E \sqsubseteq T(F)$, so that $T(E) \sqsubseteq T^2(F)$, and let $G \in \mathbf{N}$ be such that $G+E \in \mathbf{U}$ and $G+F \notin \mathbf{U}$. Then, $T(G+E) = T(G)+T(E) \sqsubseteq T(G)+T^2(F) = T(G)+T(F) = T(G+F)$. But since $T(G+E) \in \mathbf{U}$ and $\mathbf{U}$ is an upper ideal, we must have that $T(G+F) \in \mathbf{U}$, a contradiction, since $G+F \notin \mathbf{U}$ and T strongly preserves $\mathbf{U}$. Thus, $E \not\sqsubseteq T(F)$.

3. Preservers of primitivity for Boolean symmetric matrices.

In this section, we let N denote the subsemimodule of $S_n(\mathbb{B})$ generated by all digons and the single diagonal cell, $E_{1,1}$. Clearly, $S_n^{(0)}(\mathbb{B}) \subseteq N \subseteq S_n(\mathbb{B})$. Let $B \in S_n(\mathbb{B})$. By the expression $B > O$ we shall mean that $b_{i,j} \neq 0$ for all $1 \le i,j \le n$. A matrix $A \in S_n(\mathbb{B})$ is *primitive* if $A^k > O$ for some exponent k. The *exponent* of a primitive matrix A is the minimum k such that $A^k > O$, and we denote this $\exp(A)$. Let $U_p = \{A \in N \mid A$ is primitive in $S_n(\mathbb{B})\}$

Lemma 3.1. *U_p is an upper ideal in N.*

Proof. Suppose $A \in U_p$. Then, for some positive integer k, $A^k > O$. Let $X \in N$. Then $(A+X)^k = A^k + Y$ for some $Y \in S_n(\mathbb{B})$, and since $A^k > O$, it follows that $A^k + Y > O$, and hence $(A+X)$ is primitive, so that $A+X \in U_p$.

Lemma 3.6 If $n \geq 5$ *and* T: $\boldsymbol{S}_n(\mathbb{B}) \rightarrow \boldsymbol{S}_n(\mathbb{B})$ strongly preserves primitive matrices then $T(\boldsymbol{S}_n^{(0)}(\mathbb{B})) \subseteq \boldsymbol{S}_n^{(0)}(\mathbb{B})$ and $T(\boldsymbol{D}_n) \subseteq \boldsymbol{D}_n$. Further $T|\boldsymbol{D}_n$ is nonsingular.

Theorem 3.7. *Let* $n \geq 5$ *and* T: $\boldsymbol{S}_n(\mathbb{B}) \rightarrow \boldsymbol{S}_n(\mathbb{B})$ *be a linear operator. Then, T strongly preserves the set of primitive matrices in* $\boldsymbol{S}_n(\mathbb{B})$ *if and only if there exist a permutation matrix P and a nonsingular linear operator* $L: \boldsymbol{D}_n \rightarrow \boldsymbol{D}_n$ *such that* $T(X) = P(X \circ K)P^t + L(X \circ I)$ *for all* $X \in \boldsymbol{S}_n(\mathbb{B})$.

By the following examples, we have that the above theorem is not true if $n = 3, 4$.

Example 3.8 *Let* $T: \mathcal{S}_3(\mathbb{B}) \to \mathcal{S}_3(\mathbb{B})$ *be defined by* $T(D_{1,2}) = I_3$, $T(E_{i,i}) = D_{1,2}$, $T(D_{1,3}) = D_{1,3}$, *and* $T(D_{2,3}) = D_{2,3}$. *So that*

$$\begin{bmatrix} d_1 & a & b \\ a & d_2 & c \\ b & c & d_3 \end{bmatrix} \quad \to \quad \begin{bmatrix} a & d_1 + d_2 + d_3 & b \\ d_1 + d_2 + d_3 & a & c \\ b & c & a \end{bmatrix}.$$

It is easily checked that T *is a linear operator that strongly preserves primitive matrices.*

Example 3.9. *Let* T: $S_4(\mathbb{B}) \to S_4(\mathbb{B})$ *be defined by* $T(D_{1,2}) = I_4$, $T(E_{i,i}) = D_{1,3}$, $T(D_{1,3}) = D_{3,4}$, $T(D_{1,4}) = D_{1,2}$ $T(D_{2,3}) = D_{1,4}$ $T(D_{2,4}) = D_{2,3}$ *and* $T(D_{3,4}) = D_{2,4}$. *So that*

$$\begin{bmatrix} g_1 & a & b & c \\ a & g_2 & d & e \\ b & d & g_3 & f \\ c & e & f & g_4 \end{bmatrix} \rightarrow \begin{bmatrix} a & c & g_1+g_2+g_3+g_4 & d \\ c & a & e & f \\ g_1+g_2+g_3+g_4 & e & a & b \\ d & f & b & a \end{bmatrix}$$

It is easily checked that T *is a linear operator that strongly preserves primitive matrices.*

Proposition 3.10. *Up to permutational similarity on* $\boldsymbol{S}_n^{(0)}(\mathbb{B})$ *and nonsingular mappings of the diagonal matrices, the above examples are the only linear operators that strongly preserve primitivity that do not have the format of Theorem 3.7.*

4. Preservers of primitive symmetric Boolean matrices of exponent 2.

Lemma 4.1. *Let* T: $\boldsymbol{S}_n(\mathbb{B}) \rightarrow \boldsymbol{S}_n(\mathbb{B})$ *be a linear operator. If* $T(X) = O$ *for some nonzero* X *then* $T(J) = T(J \setminus E)$ *for some base element* E.

Proof. Suppose that $T(X) = O$ and E is a base element, $E \sqsubseteq X$. Then $T(E) = O$, and $T(J) = T((J \setminus E) + E) = T(J \setminus E) + T(E) = T(J \setminus E)$.

Theorem 4.6 *Let* $n \geq 4$ *and* T: $\boldsymbol{S}_n(\mathbb{B}) \rightarrow \boldsymbol{S}_n(\mathbb{B})$ *be a linear operator. Then, T strongly preserves the set of primitive symmetric matrices of exponent 2, if and only if T is a* (P,P^t)*-operator.*

Corollary 4.7 *Let* $n \geq 4$ *and* T: $\boldsymbol{S}_n(\mathbb{B}) \rightarrow \boldsymbol{S}_n(\mathbb{B})$ *be a linear operator. Then,* $T(J) = J$ *and T preserves the set of primitive symmetric matrices of exponent 2, if and only if T is a* (P,P^t)*-operator.*

Corollary 4.8 *Let* $n \geq 4$ *and* T: $\boldsymbol{S}_n(\mathbb{B}) \rightarrow \boldsymbol{S}_n(\mathbb{B})$ *be a linear operator. Then, Then, T preserves the set of primitive symmetric matrices of exponents 1 and 2, if and only if T is a* (P,P^t)*-operator.*

Corollary 4.9 *Let* $n \geq 4$ *and* T: $\boldsymbol{S}_n(\mathbb{B}) \rightarrow \boldsymbol{S}_n(\mathbb{B})$ *be a linear operator. Then, Then, T preserves the set of primitive symmetric matrices of exponent 2 and T preserves the set of symmetric matrices of Boolean rank 1, if and only if T is a* (P,P^t)*-operator.*

5. Preservers of primitive symmetric matrices over semirings

Let $\mathbb{S}$ be any commutative semiring without zero divisors. Let the set $\boldsymbol{P}_{\mathbb{S}}$ denote the set of all elements of $\mathbb{S}$ that can be written as a finite sum of squares from $\mathbb{S}$. Then, $\boldsymbol{P}_{\mathbb{S}}$ is a subsemiring of $\mathbb{S}$ since, a sum of elements of $\boldsymbol{P}_{\mathbb{S}}$ is obviously a sum of squares and if $a = a_1^2 + a_2^2 + ... + a_k^2$ and $b = b_1^2 + b_2^2 + ... + b_l^2$, then $ab = \sum_{i=1}^{k}\sum_{j=1}^{l}(a_i b_j)^2$. The semiring $\boldsymbol{P}_{\mathbb{S}}$ is called the *positive subsemiring* of $\mathbb{S}$.

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}_+$ the nonnegative reals.

Definition 5.1 A commutative semiring, $\mathbb{S}$, without zero divisors is *formally real* if and only if $\boldsymbol{P}_{\mathbb{S}}$ is antinegative.

Examples of formally real semirings include any subsemiring of the reals such as the set of all nonnegative reals $\mathbb{R}_{+}$, polynomial rings over $\mathbb{R}$, chain semirings, etc.

If B is a matrix with entries in an antinegative semiring, the notation B > O means that $b_{i,j} \neq 0$ for all i,j.

Definition 5.2. Let $\mathbb{S}$ be a formally real semiring. A matrix $A \in M_n(\mathbb{S})$ (or specifically $S_n(\mathbb{S})$) is *primitive* if and only if $A \in M_n(P_{\mathbb{S}})$ (or $A \in S_n(P_{\mathbb{S}})$) and $A^k > O$ for some positive integer k. The *exponent* of A, exp(A), is the minimum k such that $A^k > O$.

The size of entries in a matrix does not change whether or not that matrix is primitive, that is, only the location of the nonzero entries determine the primitivity of a matrix. We state the following without proof.

Theorem 5.3. Let $\mathbb{S}$ be a formally real semiring, $n \geq 5$, T: $\mathcal{S}_n(\mathbb{S}) \rightarrow \mathcal{S}_n(\mathbb{S})$ be a linear operator. Then, T strongly preserves the set of primitive matrices in $\mathcal{S}_n(\mathbb{S})$ if and only if there exist a permutation matrix P and a nonsingular linear operator $L: \mathcal{D}_n \rightarrow \mathcal{D}_n$ with $L(\mathcal{D}_n \cap \mathcal{S}_n(\mathcal{P}_{\mathbb{S}})) \subset \mathcal{S}_n(\mathcal{P}_{\mathbb{S}})$, such that $T(X) = P(X \circ B \circ K)P^t + L(X \circ I)$ for all $X \in \mathcal{S}_n(\mathbb{S})$ where $B \in \mathcal{S}_n(\mathcal{P}_{\mathbb{S}})$ has no zero entries.

We specifically state the real case:

Corollary 5.4. Let $n \geq 5$, T: $\boldsymbol{S}_n(\mathbb{R}) \rightarrow \boldsymbol{S}_n(\mathbb{R})$ be a linear operator. Then, T strongly preserves the set of primitive matrices in $\boldsymbol{S}_n(\mathbb{R})$ if and only if there exist a permutation matrix P and a nonsingular linear operator $L : \boldsymbol{D}_n \rightarrow \boldsymbol{D}_n$ with $L(\boldsymbol{D}_n \cap \boldsymbol{S}_n(\mathbb{R}_+)) \subset \boldsymbol{S}_n(\mathbb{R}_+)$, such that $T(X) = P(X \circ B \circ K)P^t + L(X \circ I)$ for all $X \in \boldsymbol{S}_n(\mathbb{R})$ where $B \in \boldsymbol{S}_n(\mathbb{R}_+)$ has no zero entries.

Theorem 5.5. Let $\mathbb{S}$ be a formally real semiring, $n \geq 4$, T: $\boldsymbol{S}_n(\mathbb{S}) \rightarrow \boldsymbol{S}_n(\mathbb{S})$ be a linear operator. Then, T strongly preserves the set of primitive symmetric matrices of exponent *2* if and only if T is a (P, P^t, B)-operator, that is, there is a permutation matrix P such that $T(X) = P(X \circ B)P^t$ for all $X \in \boldsymbol{S}_n(\mathbb{S})$ where $B \in \boldsymbol{S}_n(\boldsymbol{P}_{\mathbb{S}})$ has no zero entries.

We also state the real case here.

Corollary 5.6. Let $n \geq 4$, T: $\mathcal{S}_n(\mathbb{R}) \rightarrow \mathcal{S}_n(\mathbb{R})$ be a linear operator. Then, T strongly preserves the set of primitive symmetric matrices of exponent 2 if and only if T is a (P, P^t, B)-operator, that is, there is a permutation matrix P such that $T(X) = P(X \circ B)P^t$ for all $X \in \mathcal{S}_n(\mathbb{R})$ where all entries of B are strictly positive.

Note that the hypotheses above require that T strongly preserve the set. The following example shows that this is necessary.

Example 5.7. Let $B \in \mathcal{S}_n(\mathbb{R})$ be primitive. Define $L : \mathcal{S}_n(\mathbb{R}) \rightarrow \mathcal{S}_n(\mathbb{R})$ by $L(X) = (\sum_{i,j=1}^{n} x_{i,j})B$. Then, L is a linear operator that preserves primitivity (since the only entries in a primitive matrix are nonnegative). But clearly, L does not strongly preserve primitivity or have the form given in Theorem 5.3. If B has exponent k, then L preserves the set of primitive matrices of exponent k.

6. Further Researches.

1. Let $\mathbb{S}$ be a formally real semiring, $n \geq 4$, T: $\mathcal{S}_n(\mathbb{S}) \rightarrow \mathcal{S}_n(\mathbb{S})$ be a linear operator. Then, T preserves the set of primitive symmetric matrices of exponent any two h and k if and only if T is a (P, P^t, B)-operator.

2. Let $n \geq 4$, T: $\mathcal{S}_n(\mathbb{S}) \rightarrow \mathcal{S}_n(\mathbb{S})$ be a linear operator. Then, T strongly preserves the set of primitive symmetric matrices of exponent any $k \geq 2$ if and only if T is a (P, P^t, B)-operator.

10. References

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Thank you for your attention!