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Strong 3-commutativity preserving maps on prime rings

Xiaofei Qi

(with Prof. Jinchuan Hou)

Department of Mathematics, Shanxi University
qixf1981@sxu.edu.cn

1 Introduction

- Let $\mathcal{R}$ be an associative ring.
- For any $x, y \in \mathcal{R}$, we denote the Lie product $[x, y] = xy - yx$ the commutator of x and y.
- Recall that a map $f : \mathcal{R} \to \mathcal{R}$ preserves commutativity if $[f(a), f(b)] = 0$ whenever $[a, b] = 0$ for all a, $b \in \mathcal{R}$; preserves strong commutativity if $[f(a), f(b)] = [a, b]$ for all $a, b \in \mathcal{R}$.
- Clearly,

$$\{\text{strong commutativity preserving maps}\} \subsetneq \{\text{commutativity preserving maps}\}.$$

- The problem of characterizing commutativity preserving maps had been studied intensively on various rings and algebras.

[B1] M. Brešar, Commuting traces of biadditive mappings, commutativity preserving mappings, and Lie mappings, Trans. Amer. Math. Soc., 335 (1993), 525-546.

[B2] M. Brešar, C. R. Miers, Commutativity preserving mappings of von Neumann algebras, Canad. J. Math., 45 (1993), 695-708.

[L] Y.-F. Lin, Commutativity preserving maps on Lie ideals of prime algebras, Linear Algebra Appl., 371 (2003), 361-368.

[S] P. Šemrl, Commutativity preserving maps, Linear Algebra Appl., 429 (2008), 1051-1070.

- For strong commutativity preserving (SCP, briefly) maps, Bell and Daif in

  [BD] H. E. Bell, M. N. Daif, On commutativity and strong commutativity preserving maps, Canad. Math. Bull., 37 (1994), 443-447.

  first proved that $\mathcal{R}$ must be commutative if $\mathcal{R}$ is a prime ring and $\mathcal{R}$ admits a nonzero derivation or a non-identity endomorphism which are SCP on a nonzero right ideal of $\mathcal{R}$.

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- Brešar and Miers in

  [BM] M. Brešar, C. R. Miers, Strong commutativity preserving maps of semiprime rings, Canad. Math. Bull., 37 (1994), 457-460.

  proved that every additive SCP map f on a unital semiprime ring $\mathcal{R}$ has the form $f(a) = \lambda a + \mu(a)$, where $\lambda \in \mathcal{C}$ (the extended centroid of $\mathcal{R}$) with $\lambda^2 = 1$ (the unit of the ring) and $\mu : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ is an additive map.

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- Lin and Liu

  [LL] J. S. Lin, C. K. Liu, Strong commutativity preserving maps on Lie ideals, Lin. Alg. Appl., 428 (2008), 1601-1609.

  extended this result to Lie ideals in case the ring $\mathcal{R}$ is prime. More precisely, they proved that, if $\mathcal{L}$ is a non-central Lie ideal of a prime ring $\mathcal{R}$, then every additive SCP map $f : \mathcal{L} \to \mathcal{R}$ is of the form $f(a) = \lambda a + \mu(a)$, where $\lambda \in \mathcal{C}$ with $\lambda^2 = 1$ and $\mu : \mathcal{L} \to \mathcal{Z}(\mathcal{R})$ is an additive map, unless char$\mathcal{R} = 2$ and $\mathcal{R}$ satisfies the standard identity of degree 4.

- Qi and Hou in

[QH] X.-F. Qi, J.-C. Hou, Nonlinear strong commutativity preserving maps on prime rings, Commu. Algebras, 38 (2010), 2790-2796.

discussed general SCP maps on prime rings. If $\mathcal{R}$ is a unital prime ring with a nontrivial idempotent, then a surjective map $f : \mathcal{R} \to \mathcal{R}$ is SCP if and only if $f(a) = \alpha a + \mu(a)$ for all $a \in \mathcal{R}$, where $\alpha \in \{1, -1\}$ and $\mu : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ is an arbitrary map.

- Recently, Lee and Wong in

[LW] T. K. Lee, T. L. Wong, Nonadditive strong commutativity preserving maps, Commu. Algebra, 40 (2012), 2213-2218.

discussed the general SCP maps on a noncentral Lie ideal of prime rings and generalized the corresponding results in [LL, QH].

- For other results about SCP maps, see [DA,FS,Liu,YWW] and the references therein.

[DA] Q. Deng, M. Ashraf, On strong commutativity preserving maps, Results Math., 30 (1996), 259-263.

[FS] V. De Filippis, G. Scudo, Strong commutativity and Engel condition preserving maps in prime and semiprime rings, Linear and Multilinear Algebra, 61 (2013), 917-938.

[Liu] C. K. Liu, Strong commutativity preserving maps on subsets of matrices that are not closed under addition, Linear Algebra Appl., 458 (2014), 280-290.

[YWW] H. Yuan, Y. Wang, Y. Wang, Y. Du, Strong commutativity preserving generalized derivations on triangular rings, Oper. Matrices, 8 (2014), 773-783.

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- k-commutator: For any elements $a, b \in \mathcal{R}$, we set $[a, b]_0 = a$, $[a, b]_1 = ab - ba$, and inductively $[a, b]_k = [[a, b]_{k-1}, b]$, where k is a positive integer.

- Thus, we can introduce the concept of strong k-commutativity preserving maps.

- A map $f : \mathcal{R} \to \mathcal{R}$ is said to be strong k-commutativity preserving if $[f(a), f(b)]_k = [a, b]_k$ for all $a, b \in \mathcal{R}$.

- Obviously, strong k-commutativity preserving maps are usual SCP maps if $k = 1$.

- So, a natural problem is how to characterize strong k-commutativity preserving maps for $k > 1$.

- Qi first tried to characterize such maps on prime rings for the case $k = 2$ in [Q] and showed that a surjective map on a unital prime ring with characteristic not 2 and containing a nontrivial idempotent is strong 2-commutativity preserving if and only if it has the form $x \mapsto \lambda x + \mu(x)$, where λ is an element in the extended centroid of the ring satisfying $\lambda^3 = 1$ and μ is a central valued map.

[Q] X.-F. Qi, Strong 2-commutativity preserving maps on prime rings, Publ. Math. Debrecen, 88(1-2) (2016), 119-129.

- Note that,

$$[a,b]_k = \sum_{i=1}^{k} (-1)^i C_k^i b^i a b^{k-i}, \qquad (1.1)$$

where $C_k^i = \frac{k(k-1)\cdots(i+2)(i+1)}{i(i-1)\cdots 2\cdot 1}$.

- So, with k increasing, the problem of characterizing strong k-commutativity preserving maps becomes much more difficult.

- Let X be a real or complex Banach space with $\dim X \geq 2$, $\mathcal{A}$ be a standard operator algebra on X (a subalgebra of bounded linear operators which contains the identity I and all finite rank operators).
- Let $\Phi : \mathcal{A} \to \mathcal{A}$ be a map with range containing all operators of rank ≤ 1.

- In

  [LH1] M.-Y. Liu, J.-C. Hou, Strong 3-commutativity preserving maps on standard operator algebras, arXiv: 1601.06336v1,

  it is shown that Φ is strong 3-commutativity preserving if and only if there exist a (nonlinear) functional h and a scalar λ with $\lambda^4 = 1$ such that $\Phi(A) = \lambda A + h(A)I$ for all $A \in \mathcal{A}$.

- For the case when X is complex, we proved in

  [HQ] J.-C. Hou, X.-F. Qi, Strong k-commutativity preservers on complex standard operator algebras, submitted,

  that, for any positive integer k, Φ is strong k-commutativity preserving if and only if $\Phi(A) = \lambda A + h(A)I$ for all $A \in \mathcal{A}$ with $\lambda^{k+1} = 1$.

- **Remark.** The approach in [HQ, LH1] strongly depends on the existence of rank one and rank two idempotents in standard operator algebras as well as the spectral analysis. So the methods used there are not valid for general prime rings.

- It was conjectured in [LH1] that:

  every strong k-commutativity preserving surjective map on a unital prime ring has the form $x \mapsto \lambda x + \mu(x)$, where λ is an element in the extended centroid of the ring with $\lambda^{k+1} = 1$ and μ is a map from the ring into its extended centroid.

- We have no idea how to prove this conjecture. But in this talk, we give an affirmative answer for $k = 3$.

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2 Main result and corollaries

- The following is our main result in this talk.

Theorem 2.1. *Let $\mathcal{R}$ be a unital prime ring with characteristic not 2 and containing a nontrivial idempotent. Assume that $f : \mathcal{R} \to \mathcal{R}$ is a surjective map. Then f is strong 3-commutativity preserving, that is,*

$$[f(x), f(y)]_3 = [x, y]_3 \text{ for all } x, y \in \mathcal{R}$$

if and only if there exist a map $\mu : \mathcal{R} \to \mathcal{C}$ and an element $\lambda \in \mathcal{C}$ with $\lambda^4 = 1$ such that

$$f(x) = \lambda x + \mu(x) \text{ for all } x \in \mathcal{R},$$

where $\mathcal{C}$ is the extended centroid of $\mathcal{R}$.

a key lemma:

Lemma. *Let* $\mathcal{R}$ *be a unital prime ring with characteristic not* $2, 3$ *and* $s \in \mathcal{R}$. *Then*

$$[x, s]_3 = 0 \ \text{ holds for all } \ x \in \mathcal{R}$$

if and only if there exist $c \in \mathcal{C}$ *and* $n \in \mathcal{N}_{\mathcal{Q}}$ *such that*

$$s = c + n,$$

where $\mathcal{Q}$ *and* $\mathcal{C}$ *are respectively the the maximal left ring of quotients and the extended centroid of* $\mathcal{R}$.

Applications

- It is shown in

  [A] P. Ara, The extended centroid of C*-algebras, Arch. Math., 54 (1990), 358-364,

  that, if $\mathcal{A}$ is a prime C*-algebra, then its extended centroid $\mathcal{C}(\mathcal{A}) = \mathbb{C}$.

- So, by Theorem 2.1, the following corollary is immediate.

  Corollary 2.2. *Let $\mathcal{A}$ be a unital prime C*-algebra containing a nontrivial idempotent and $\Phi : \mathcal{A} \to \mathcal{A}$ a surjective map. Then Φ preserves strong 3-commutativity if and only if there exists a complex number $\lambda \in \{\pm 1, \pm i\}$ and a functional $g : \mathcal{A} \to \mathbb{C}$ such that $\Phi(A) = \lambda A + g(A)I$ for all $A \in \mathcal{A}$.*

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- Recall that a von Neumann algebra $\mathcal{M}$ is a C*-subalgebra of some $\mathcal{B}(H)$, the algebra of all bounded linear operators acting on a complex Hilbert space H, which satisfies the double commutant property: $\mathcal{M}'' = \mathcal{M}$, where $\mathcal{M}' = \{T : T \in \mathcal{B}(H) \text{ and } TA = AT \ \forall A \in \mathcal{M}\}$ and $\mathcal{M}'' = \{\mathcal{M}'\}'$ ([KR]).

- $\mathcal{M}$ is called a factor if its center $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$.

- It is well-known that von Neumann algebras are unital containing many nontrivial idempotents, and every factor von Neumann algebra is prime.

[KR] R. V. Kadison, J. R. Ringrose, Fundamentals of the Theory of Operator Algebras, Vol. I, Academic Press, New York, 1983, Vol. II, Academic Press, New York, 1986.

- So the following corollary is obvious by Corollary 2.3.

Corollary 2.3. *Let $\mathcal{M}$ be a factor von Neumann algebra. Assume that $\Phi : \mathcal{M} \to \mathcal{M}$ is a surjective map. Then Φ is strong 3-commutativity preserving if and only if there exist a functional $g : \mathcal{M} \to \mathbb{C}$ and a scalar $\alpha \in \{\pm 1, \pm i\}$ such that $\Phi(A) = \alpha A + g(A)I$ for all $A \in \mathcal{M}$.*

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- Since every standard operator algebra is a prime ring containing many idempotents, and the center of a standard algebra is $\mathbb{F}I$, by Theorem 2.1, we can also get the following result.

Corollary 2.4. *Let X be a Banach space over the real or complex field $\mathbb{F}$ with $\dim X \geq 2$ and $\mathcal{A} \subseteq \mathcal{B}(X)$ be a standard operator algebra. Assume that $\Phi : \mathcal{A} \to \mathcal{A}$ is a surjective map. Then Φ is strong 3-commutativity preserving if and only if there exist a functional $h : \mathcal{A} \to \mathbb{F}$ and a scalar $\lambda \in \mathbb{F}$ with $\lambda^4 = 1$ such that $\Phi(A) = \lambda A + h(A)I$ for all $A \in \mathcal{A}$.*

Thank You for Your Attention!