From Hlawka Inequality to Generalized Matrix Functions

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Joint work with
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Outline

- Parallelogram identity and extension

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- Further extensions

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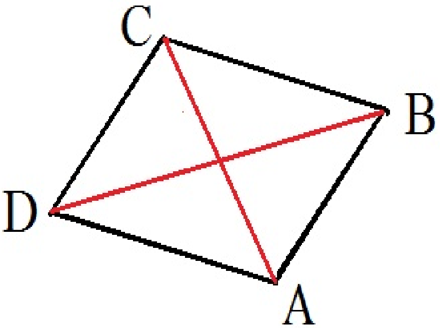

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& |AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 \\
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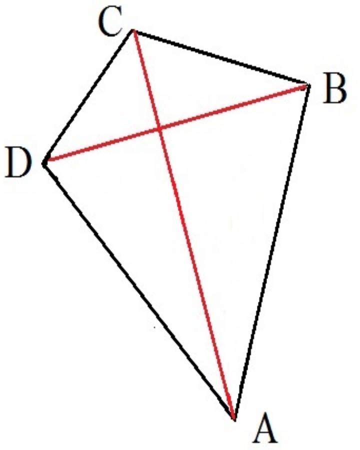
Extension of parallelogram identity

For a general quadrilateral $ABCD$, the sum of squares of the four sides is **greater than or equal to** the sum of squares of the two diagonals.



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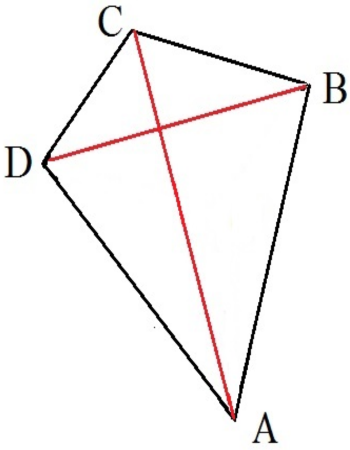

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Equality holds if and only if

$ABCD$ **is a parallelogram.**

Quadrilateral identity

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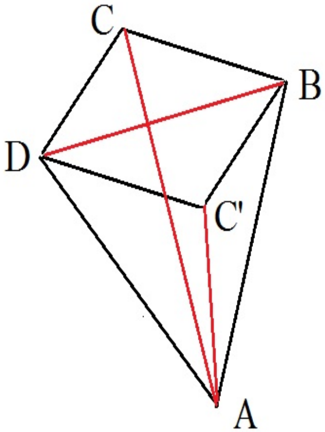

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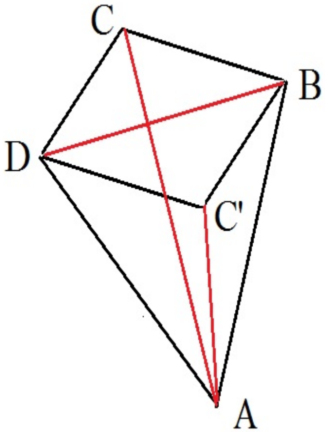

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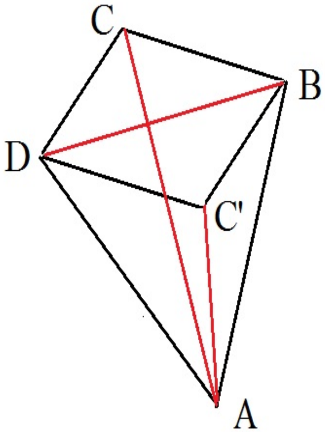

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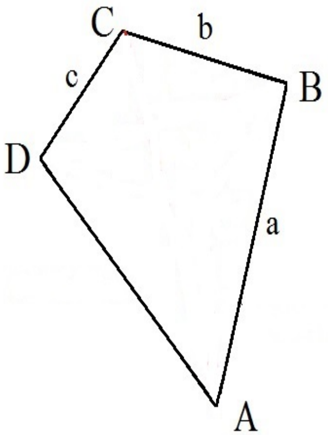
Hlawka inequality



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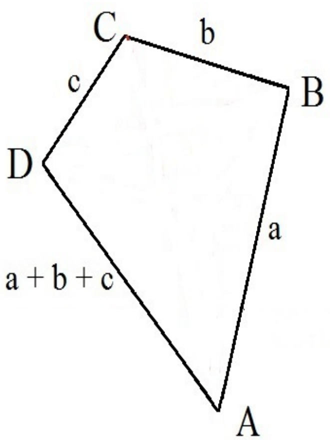

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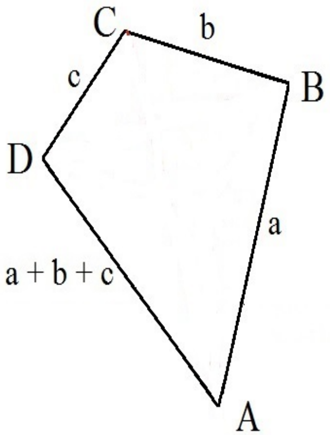

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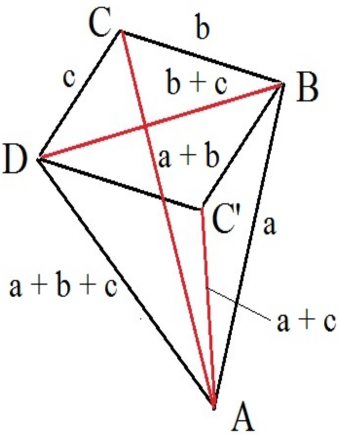

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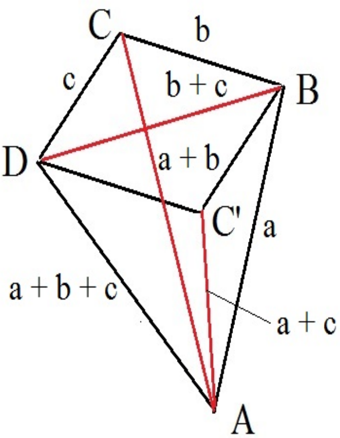

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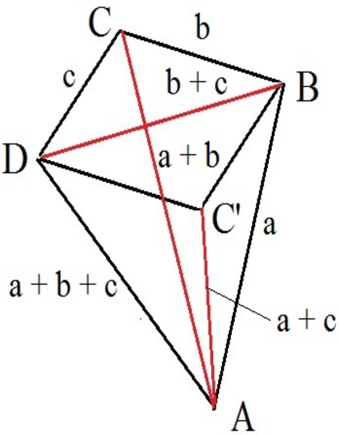

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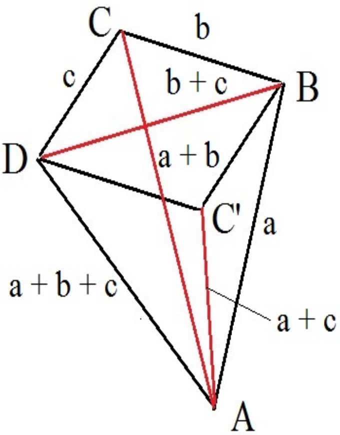

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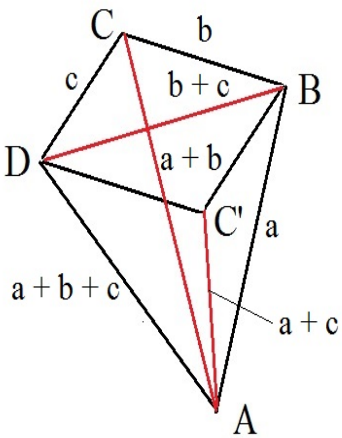

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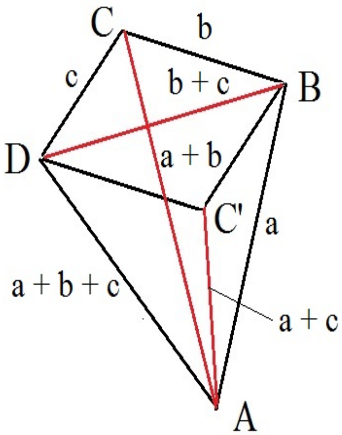

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E.g. $\mathbf{a} = (1, 1, -1)$, $\mathbf{b} = (1, -1, 1)$,

$\mathbf{c} = (-1, 1, 1)$.

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Compare the Hlawka inequality for **a**, **b**, **c** :

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if $G = S_n$, $\chi(\sigma) = 1$, then $d_\chi(A) = \mathrm{per}(A)$.

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For example, let $A = (a_{ij}) \in M_2$. Then

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$$v_1^t(A \otimes A)v_1 = a_{11}a_{22}, \quad v_2^t(A \otimes A)v_2 = \det(A), \quad v_3^t(A \otimes A)v_3 = \mathrm{per}(A).$$

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$$(A+B)^{\otimes m} = S_{m,0}(A,B) + S_{m,1}(A,B) + \cdots + S_{m,m-1}(A,B) + S_{m,m}(A,B),$$

where $S_m(A,B)$ is the sum of $X_1 \otimes \cdots \otimes X_m$ such that $m-k$ of the X_j's are A and the rest are B.

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- For example, for $m = 3$,
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- If A and B are positive semidefinite, then $(A+B)^{\otimes m} \geq A^{\otimes m} + B^{\otimes m}$.
- It follows immediately that for any generalized matrix function d_χ,

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- For example, we have $\prod_{j=1}^n (A_{jj}+B_{jj}) \geq \prod_{j=1}^n A_{jj} + \prod_{j=1}^n B_{jj}$,
$$\det(A+B) \geq \det(A) + \det(B), \quad \mathrm{per}(A+B) \geq \mathrm{per}(A) + \mathrm{per}(B).$$

Hlawka inequality of generalized matrix functions

Theorem 1 (W. Berndt and S. Sra, 2015)

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Then use $d_\chi(X) = \mathbf{v}_\chi^* \left(X^{\otimes n} \right) \mathbf{v}_\chi$ for $X \in M_n$.

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For $m = 4$, (1) is equivalent to

$$\begin{aligned} & d(A_1 + A_2 + A_3 + A_4) + d(A_1 + A_2) + d(A_1 + A_3) + \cdots + d(A_3 + A_4) \\ \geq \quad & d(A_1 + A_2 + A_3) + \cdots + d(A_2 + A_3 + A_4) + d(A_1) + \cdots + d(A_4) \end{aligned}$$

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4) The restriction of $r \in \{1, \ldots, m-2\} \cup [m-1,\ \infty)$ is necessary.

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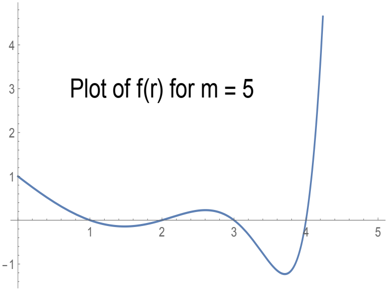

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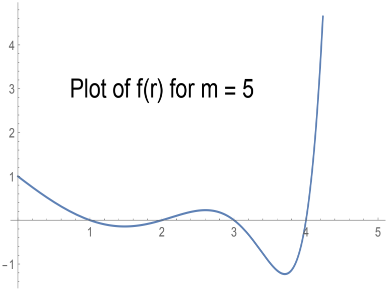

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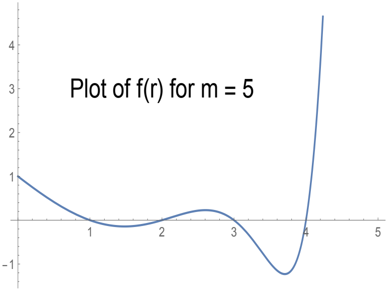

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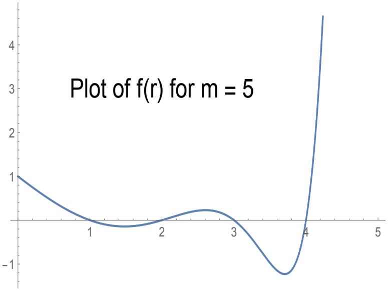

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Further extensions

Theorem 4

Suppose $A_1, \ldots, A_m \in M_n$ are positive semi-definite matrices, and $r \in \{1\} \cup [2, \infty)$. Let $K = \{1, \ldots, m\}$ and $K_j = \{J \leq K : |J| = j\}$ for $1 \leq j \leq m$. For each $J \leq K$, let $A_J = \sum_{j \in J} A_j$. Then for every $1 \leq k < \ell < p \leq m$ and any generalized matrix function $d(A)$, we have

$$\frac{\ell - k}{p \begin{pmatrix} m \\ p \end{pmatrix}} \sum_{J \in K_p} d\left(A_J\right)^r + \frac{p - \ell}{k \begin{pmatrix} m \\ k \end{pmatrix}} \sum_{J \in K_k} d\left(A_J\right)^r \geq \frac{p - k}{\ell \begin{pmatrix} m \\ \ell \end{pmatrix}} \sum_{J \in K_\ell} d\left(A_J\right)^r .$$

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Remark Theorem 2 (2) corresponds to $r = k = 1, \ \ell = 2, \ p = m$ in the above Theorem.

Further extensions

Proposition 5

Suppose $A_i, B_i, C_i \in M_{n_i}$ are positive semi-definite matrices for $1 \leq i \leq k$. Then

$$\begin{aligned} & \otimes_{i=1}^k (A_i + B_i + C_i) + \otimes_{i=1}^k A_i + \otimes_{i=1}^k B_i + \otimes_{i=1}^k C_i \\ \geq \quad & \otimes_{i=1}^k (A_i + B_i) + \otimes_{i=1}^k (A_i + C_i) + \otimes_{i=1}^k (B_i + C_i). \end{aligned}$$

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Consequently, we have

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Further extensions

Proposition 5

Suppose $A_i, B_i, C_i \in M_{n_i}$ are positive semi-definite matrices for $1 \leq i \leq k$. Then

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$$\begin{aligned} & \otimes_{i=1}^k d_i(A_i + B_i + C_i) + \otimes_{i=1}^k d_i(A_i) + \otimes_{i=1}^k d_i(B_i) + \otimes_{i=1}^k d_i(C_i) \\ \ge \quad & \otimes_{i=1}^k d_i(A_i + B_i) + \otimes_{i=1}^k d_i(A_i + C_i) + \otimes_{i=1}^k d_i(B_i + C_i). \end{aligned}$$

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Therefore, we can replace $d_\chi(X)$ in Theorem 1 by a product of generalized matrix functions $d(X_1, \ldots, X_k) = d_1(X_1) \cdots d_k(X_k)$.

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$$\begin{aligned} & [\det(A_1 + B_1 + C_1)\mathrm{per}\,(A_1 + B_1 + C_1)]^r \\ & \quad + [\det(A_1)\mathrm{per}\,(A_2)]^r + [\det(B_1)\mathrm{per}\,(B_2)]^r + [\det(C_1)\mathrm{per}\,(C_2)]^r \\ \geq \; & [\det(A_1 + B_1)\mathrm{per}\,(A_2 + B_2)]^r + [\det(A_1 + C_1)\mathrm{per}\,(A_2 + C_2)]^r \\ & \quad + [\det(B_1 + C_1)\mathrm{per}\,(B_2 + C_2)]^r. \end{aligned}$$

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Theorem 3 and Theorem 4 can also be generalized in a similar way.

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- It can be shown that all such $x_{ij}, x_{ijk}, x_{ijk\ell}$, etc are nonnegative.
- Then transform each inequality on $d(A_j)^r, d(A_i + A_j)^r, d(A_i + A_j + A_k)^r$, etc. to an inequalities of the form $f(\mathbf{x}) \geq 0$, where

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Thank you very much

for your attention!

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Iowa State University

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